# STAT 714 fa 2023

## Linear algebra review 4/6

### Orthogonal subspaces, bases, projections, Gram-Schmidt

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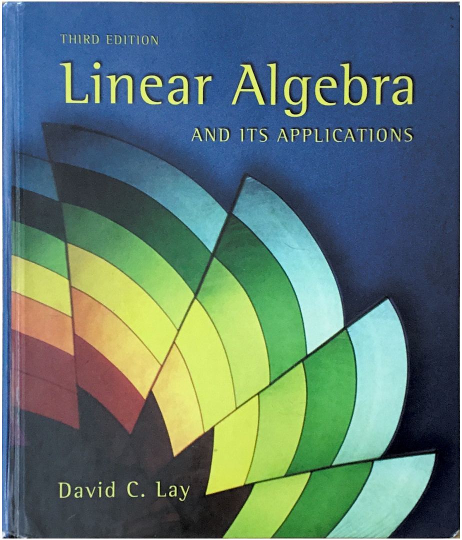
These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

These notes include scanned excerpts from Lay (2003):



## Orthogonal and orthonormal sets of vectors

A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\} \in \mathbb{R}^n$ is an *orthogonal set* if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$.

If in addition $\|\mathbf{v}_i\| = 1$ for $i = 1, \dots, n$, the set is an *orthonormal set*.

**Example:** The *elementary vectors*

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \dots \quad , \quad \mathbf{e}_{n-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

in $\mathbb{R}^n$ are an orthonormal set; moreover $\text{Span}\{\mathbf{e}_1, \dots, \mathbf{e}_n\} = \mathbb{R}^n$.

## Result (Orthonormal columns)

*An $m \times n$ matrix $\mathbf{U}$ has orthonormal columns if and only if $\mathbf{U}^T\mathbf{U} = \mathbf{I}$.*

**Prove the result.**

**Exercise:** Check if these matrices have orthonormal columns:

$$\mathbf{C} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix}$$

**Theorem (Results for matrices with orthonormal columns)**

*Let* $\mathbf{U}$ *be an* $m \times n$ *matrix with orthonormal columns and let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. *Then*

1. $\|\mathbf{U}\mathbf{x}\| = \|\mathbf{x}\|$
2. $(\mathbf{U}\mathbf{x}) \cdot (\mathbf{U}\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$
3. $(\mathbf{U}\mathbf{x}) \cdot (\mathbf{U}\mathbf{y}) = 0 \iff \mathbf{x} \cdot \mathbf{y} = 0$

**Exercise:** Prove the above results.

## Orthogonal matrix

An *orthogonal matrix* is a square invertible matrix $\mathbf{U}$ such that $\mathbf{U}^{-1} = \mathbf{U}^T$.

If $\mathbf{U}$ is $n \times n$ with orthonormal columns, we have $\mathbf{U}^T\mathbf{U} = \mathbf{I}_n$ and $\mathbf{U}\mathbf{U}^T = \mathbf{I}_n$ (since the left inverse is the right inverse).

So square matrices with orthonormal columns are called *orthogonal matrices*.

## Orthogonal complement

Let $W$ be a subspace of $\mathbb{R}^n$.

- If $\mathbf{z} \cdot \mathbf{x} = 0$ for all $\mathbf{x} \in W$, we say $\mathbf{z}$ is *orthogonal to* $W$.
- The *orthogonal complement* of $W$ is the set of all such vectors $\mathbf{z}$.

Denote the orthogonal complement of a subspace $W$ as $W^\perp$.

## Theorem (Results about orthogonal complements)

*Let $W$ be a subspace of $\mathbb{R}^n$. Then:*

1. *A vector $\mathbf{x}$ is in $W^\perp$ iff $\mathbf{x}$ is orthogonal to every vector in a set that spans $W$.*
2. *$W^\perp$ is a subspace of $\mathbb{R}^n$.*

**Prove the result.**

**Exercise:** Find the orthogonal complements of these subspaces:

1. $W_1 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$
2. $W_2 = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Draw pictures.

The next result tells us how to find an orthogonal complement.

**Theorem (Orthogonal complement of column and row space)**

*Let* $\mathbf{A}$ *be an* $m \times n$ *matrix. Then* $(\operatorname{Col}\mathbf{A})^{\perp} = \operatorname{Nul}\mathbf{A}^T$ *and* $(\operatorname{Row}\mathbf{A})^{\perp} = \operatorname{Nul}\mathbf{A}$.

**Prove the result.**

**Exercise:** Find the orthogonal complement of Span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

**Theorem (Chopping result. Cf. Monahan (2008) Result A.6)**

*For two vector spaces* $W$ *and* $V$, $W \subset V$ *implies* $V^\perp \subset W^\perp$.
*As a consequence, we have additionally* $W = V \iff W^\perp = V^\perp$.

**Prove the result.**

## Theorem (Lemma 2.1 and Results 2.2 and 2.4 of Monahan)

*For an $n \times p$ matrix $\mathbf{X}$, we have*

1. $\operatorname{Nul} \mathbf{X}^T\mathbf{X} = \operatorname{Nul} \mathbf{X}$
2. $\operatorname{Col} \mathbf{X}^T\mathbf{X} = \operatorname{Col} \mathbf{X}^T$
3. $\mathbf{X}^T\mathbf{X}\mathbf{A} = \mathbf{X}^T\mathbf{X}\mathbf{B} \iff \mathbf{X}\mathbf{A} = \mathbf{X}\mathbf{B}.$

**Prove the results.**

## Theorem (An orthogonal set of nonzero vectors is a basis)

*If* $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ *is an orthogonal set of nonzero vectors in* $\mathbb{R}^n$*, then it is linearly independent and therefore a basis for* $\text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$.

An *orthogonal basis* is a basis which is an orthogonal set.

**Prove the result.**

**Example:** Here are two bases for the same space—one orthogonal, one not:

$$\text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

How to find a vector's "coordinates" with respect to an orthogonal basis:

## Result (Find a vector's coordinates wrt an orthogonal basis)

*Let* $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ *be an orth. basis for a subspace* $W$ *of* $\mathbb{R}^n$. *Then for each* $\mathbf{y} \in W$,

$$\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p, \quad \text{where } c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}, \quad j = 1, \ldots, p.$$

**Prove the result.**

**Exercise:** Find the coefficients to construct $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ from the orthogonal basis

$$\left\{ \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix} \right\}$$

## Theorem (Orthogonal decomposition theorem)

*Let $W$ be a subspace of $\mathbb{R}^n$. Then we can decompose any $\mathbf{y} \in \mathbb{R}^n$ uniquely as*

$$\mathbf{y} = \hat{\mathbf{y}} + \hat{\mathbf{e}}, \quad \textit{where} \quad \hat{\mathbf{y}} \in W \ \ \textit{and} \ \ \hat{\mathbf{e}} \in W^\perp.$$

*Moreover, for any orthogonal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ for $W$, we have*

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\mathbf{u}_p \quad \textit{and} \quad \hat{\mathbf{e}} = \mathbf{y} - \hat{\mathbf{y}}.$$

The vector $\hat{\mathbf{y}}$ is called the *orthogonal projection of $\mathbf{y}$ onto $W$*, denoted by $\operatorname{proj}_W \mathbf{y}$.

Note that if $\mathbf{y} \in W$ then $\operatorname{proj}_W \mathbf{y} = \mathbf{y}$.

**Prove the theorem.**

## Theorem (Best approximation theorem)

*Let $W$ be a subspace of $\mathbb{R}^n$, $\mathbf{y}$ any vector in $\mathbb{R}^n$, and $\hat{\mathbf{y}} = \text{proj}_W \mathbf{y}$. Then*

$$\|\mathbf{y} - \hat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\| \quad \textit{for all } \mathbf{v} \in W \textit{ distinct from } \hat{\mathbf{y}}.$$

**Prove the result.**

**Exercise:** Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Give the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ on Span$\{\mathbf{v}\}$. Draw pictures! Check orthogonality of $\mathbf{y} - \hat{\mathbf{y}}$ and $\mathbf{y}$.

The Gram-Schmidt process turns a basis into an orthogonal basis.

## The Gram–Schmidt Process

Given a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ for a subspace $W$ of $\mathbb{R}^n$, define

$$
\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \cdots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}
\end{aligned}
$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for $W$. In addition

$$\text{Span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\,\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \qquad \text{for } 1 \leq k \leq p \tag{1}$$

**Prove the result.**

**Exercise:** Use the Gram-Schmidt process to orthogonalize the basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Lay, D. C. (2003). *Linear algebra and its applications. Third edition.* Pearson Education.

Monahan, J. F. (2008). *A primer on linear models*. CRC Press.