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Linear algebra review 5/6

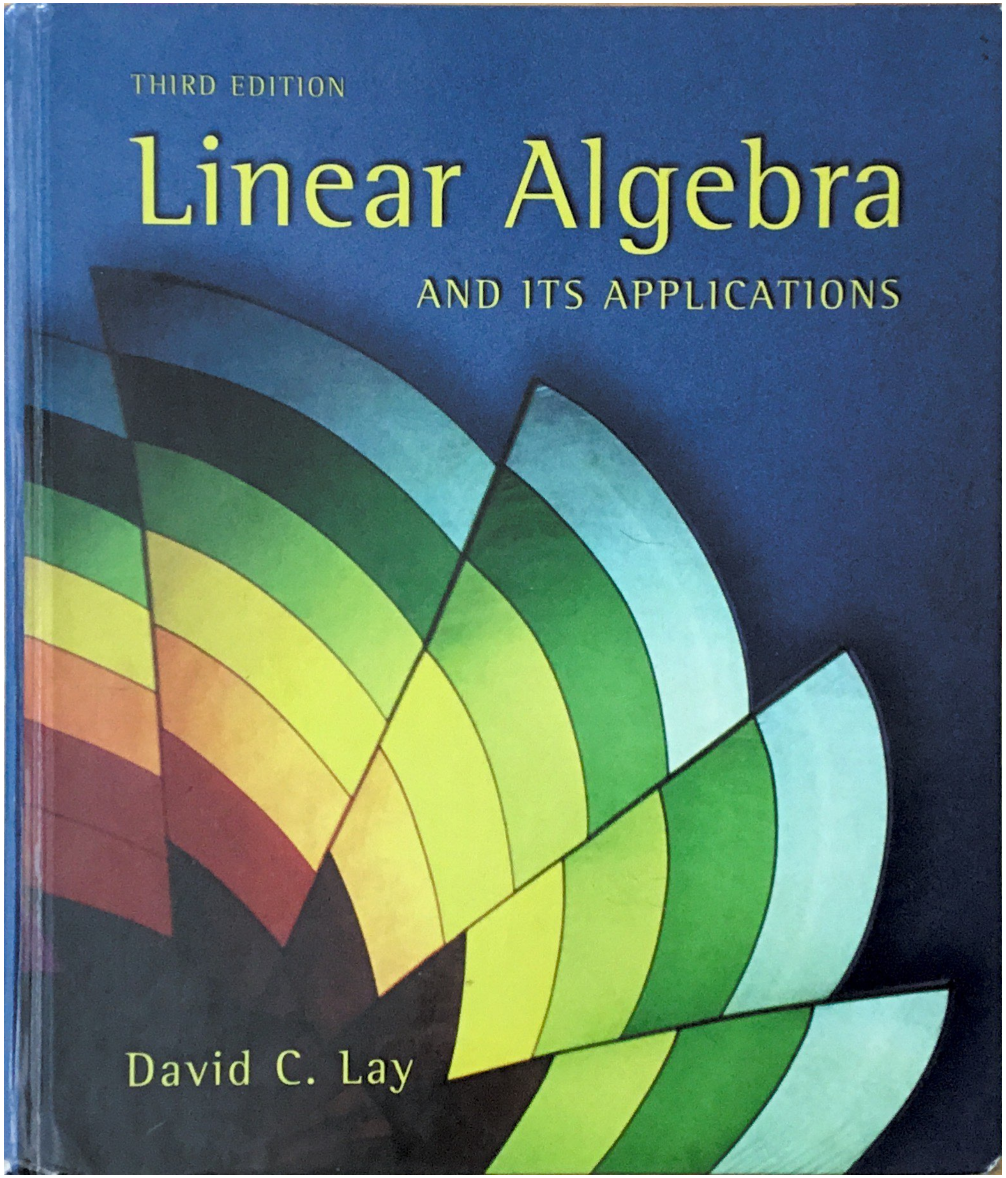
Eigenvalues and eigenvectors

Karl B. Gregory

University of South Carolina

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

These notes include scanned excerpts from Lay (2003):



Eigenvectors and eigenvalues

Let $\mathbf{A}$ be an $n \times n$ matrix.

1. An *eigenvector* of $\mathbf{A}$ is a $\boxed{\text{nonzero}}$ vec. $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.
2. A scalar λ is an *eigen~~vector~~* value of $\mathbf{A}$ if there is a nontrivial solution to $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. Such an $\mathbf{x}$ is called an *eigenvector corresponding to λ*.

Interpretation: The magnitudes of the eigenvalues of $\mathbf{A}$ represent the amount by which $\mathbf{A}$ stretches or shrinks certain vectors.

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

$$\iff A\underset{\sim}{x} - \lambda\underset{\sim}{x} = 0 \iff (A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$$

Exercise: For $\mathbf{A} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ check whether

1. the vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ are eigenvectors.
2. the values -4 and 6 are eigenvalues.

not

(1) $$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 14 \end{bmatrix} = 7 \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

(2) check if $(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$ has nontrivial solution

$$\left[\begin{pmatrix}1 & 6\\ 5 & 2\end{pmatrix} - (-4)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right] = \begin{bmatrix}5 & 6\\ 5 & 6\end{bmatrix}$$

check if $\begin{pmatrix}5 & 6\\ 5 & 6\end{pmatrix} \underset{\sim}{x} = \underset{\sim}{0}$ has non trivial solution.

The vector $\underset{\sim}{x} = \begin{bmatrix}6\\ -5\end{bmatrix}$ is a solution so

YES, -4 is an eigenvalue.

$$\boxed{\underset{\sim}{x} = \begin{pmatrix}6\\ -5\end{pmatrix} \text{ is an eigenvector of } A = \begin{pmatrix}1 & 6\\ 5 & 2\end{pmatrix} \text{ corresponding to the eigenvalue } -4}$$

Every vector in $\text{Span}\left\{\begin{pmatrix}6\\ -5\end{pmatrix}\right\}$ is also an eigenvector of A corresponding to the eigenvalue -4.

the set of all vectors $\{\underset{\sim}{x} : A\underset{\sim}{x} = \lambda \underset{\sim}{x}\}$

Eigenspaces

If λ is an eigenvalue of $\mathbf{A}$, the set of all solutions to $(\mathbf{A} - \lambda \mathbf{I}_n)\mathbf{x} = \mathbf{0}$ is called the *eigenspace* of $\mathbf{A}$ corresponding to λ.

Exercise: An eigenvalue of the matrix below is 2. Find a basis for the corresponding eigenspace:

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

Eigenspace of A corresponding to $\lambda = 2$ is

$\{\underset{\sim}{x} : A\underset{\sim}{x} = 2\underset{\sim}{x}\} = \{\underset{\sim}{x} : (A - 2I)\underset{\sim}{x} = 0\} = \text{Null}(A - 2I)$

$$[A - 2I \quad \underset{\sim}{0}] = \begin{pmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$2x_1 - x_2 + 6x_3 = 0$$

$$x_1 = \tfrac{1}{2}x_2 - 3x_3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

Eigenspace of A corresponding to $\lambda = 2$ $\to$ $\text{Span}\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$

Theorem (Linear independence of eigenspaces)

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of a matrix, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent

Prove the result.

Let $\lambda_1, \ldots, \lambda_r$ be distinct eigenvalues of A.

Let $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r$ be corresponding eigenvectors.

Suppose (by way of contradiction) that $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r$ are not lin. indep.

This means there is some $p < r$ such that $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_p$ are linearly independent and we can write $\underset{\sim}{v}_{p+1} = c_1 \underset{\sim}{v}_1 + \cdots + c_p \underset{\sim}{v}_p$.

Then

$$
\begin{aligned}
A \underset{\sim}{v}_{p+1} &= A (c_1 \underset{\sim}{v}_1 + \cdots + c_p \underset{\sim}{v}_p) \\
&= c_1 A \underset{\sim}{v}_1 + \cdots + c_p A \underset{\sim}{v}_p \\
&= c_1 \lambda_1 \underset{\sim}{v}_1 + \cdots + c_p \lambda_p \underset{\sim}{v}_p
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{p+1} \underset{\sim}{v}_{p+1} &= \lambda_{p+1} (c_1 \underset{\sim}{v}_1 + \cdots + c_p \underset{\sim}{v}_p) \\
&= c_1 \lambda_{p+1} \underset{\sim}{v}_1 + \cdots + c_p \lambda_{p+1} \underset{\sim}{v}_p
\end{aligned}
$$

Since $A \underset{\sim}{v}_{p+1} = \lambda_{p+1} \underset{\sim}{v}_{p+1}$, we have from the above

$$
c_1 \lambda_1 \underset{\sim}{v}_1 + \cdots + c_p \lambda_p \underset{\sim}{v}_p = c_1 \lambda_{p+1} \underset{\sim}{v}_1 + \cdots + c_p \lambda_{p+1} \underset{\sim}{v}_p
$$

$\Leftrightarrow$

$$
c_1 \underbrace{(\lambda_1 - \lambda_{p+1})}_{\neq 0} \underset{\sim}{v}_1 + \cdots + c_p \underbrace{(\lambda_p - \lambda_{p+1})}_{\neq 0} \underset{\sim}{v}_p = 0
$$

(remember $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_p$ are lin. dependent).

$\Rightarrow \quad c_1 = \cdots = c_p = 0.$

$\Rightarrow$

Since $\underset{\sim}{v}_{p+1} = c_1 \underset{\sim}{v}_1 + \cdots + c_p \underset{\sim}{v}_p$, then $\underset{\sim}{v}_{p+1} = \underset{\sim}{0}$.

But $\underset{\sim}{v}_{p+1}$ is an eigenvector, so it cannot be $\underset{\sim}{0}$.

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

s. The number 0 is *not* an eigenvalue of A.

t. The determinant of A is *not* zero.

$\text{Nul}\, A \neq \{0\}$

Prove the first of the above results.

Let A have eigenvalue $\lambda = 0$.

Means there exists nonzero $\underset{\sim}{x}$ such that $A\underset{\sim}{x} = 0\underset{\sim}{x} = \underset{\sim}{0}$.

$\Rightarrow$ columns of A are linearly dependent.

I.e. A does not have full-column rank.

Recall $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$

For larger matrices the determinant is defined as follows:

Definition of determinant by co-factor expansion

Let $\mathbf{A}$ be an $n \times n$ matrix with i, j entry a_{ij} and let $\mathbf{A}_{(i,j)}$ be the matrix $\mathbf{A}$ with row i and column j removed. Then, for $i, j = 1, \dots, n$, define the (i, j)-cofactor as

$$C_{ij} = (-1)^{i+j} \det \mathbf{A}_{(i,j)}.$$

Then for any i and j we have $\det \mathbf{A} = \sum_{k=1}^{n} a_{ik} C_{ik} = \sum_{k=1}^{n} a_{kj} C_{kj}$.

Often write $\det \mathbf{A}$ as $|\mathbf{A}|$.

This requires over $n!$ multiplications, so computers use a different method.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Exercise: Compute the determinant of

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}.$$

$$\det A = 1 \underbrace{\begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix}}_{-2} - 2 \begin{vmatrix} 5 & 0 \\ -2 & 0 \end{vmatrix} + 0 \begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix}$$

$$= -2$$

The determinant and cofactors give a formula for a matrix inverse:

Theorem (An inverse formula using cofactors)

If $\mathbf{A}$ *is an invertible* $n \times n$ *matrix, we have*

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}^{T}$$

Exercise: Compute the inverse of the matrix on the previous slide.

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}^{-1} = \frac{1}{(-2)} \begin{bmatrix} \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Theorem (Some properties of determinants)

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be* $n \times n$ *matrices. We have*

1. $\mathbf{A}$ *is invertible if and only if* $\det \mathbf{A} \neq 0$. ✓
2. $\det \mathbf{AB} = (\det \mathbf{A})(\det \mathbf{B})$ Memorize
3. $\det \mathbf{A}^T = \det \mathbf{A}$ ✓

Discuss (no time to prove) the above results.

Theorem (Finding eigenvalues with the characteristic equation)

A scalar λ is an eigenvalue of an $n \times n$ matrix $\mathbf{A}$ iff λ satisfies $\underbrace{\det(\mathbf{A} - \lambda\mathbf{I})}_{\text{characteristic polynomial.}} = 0$.

The equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is called the *characteristic equation*.

RHS an n-degree polynomial, which has n roots (some roots may be complex).

Understand the above result.

$$\det(A - \lambda I) = 0$$
$\iff$ $A - \lambda I$ is not invertible
$\iff$ $A - \lambda I$ has linearly dependent columns
$\iff$ $(A - \lambda I)\underset{\sim}{x} = 0$ has nontrivial solutions
$\iff$ $\exists$ nonzero $\underset{\sim}{x}$ s.t. $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$.

Exercise: Find the characteristic equation of

$\mathbf{A} =$ [redacted]

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\boxed{|A - \lambda I|} = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} + 3\begin{vmatrix} 3 & 3 \\ 3 & 1-\lambda \end{vmatrix} + 3\begin{vmatrix} 3 & 3 \\ -5-\lambda & -3 \end{vmatrix}$$

$\vdots$

$$= (1-\lambda)(2+\lambda)^2$$

$$= 0$$

Characteristic equation $\underbrace{(1-\lambda)(2+\lambda)^2}_{\text{characteristic polynomial}} = 0.$

$\lambda = -2$ has multiplicity 2

Eigenvalues are $\boxed{\lambda = 1, -2, -2}$

$$\begin{bmatrix} d_1 & & 0 \\ \cdot & d_2 & \\ \vdots \ \text{nonzero} & & \ddots \\ \cdot & \cdots & \cdot \ d_n \end{bmatrix} \qquad \begin{bmatrix} d_1 & \cdot & \cdot & \cdot \\ & d_2 & \text{nonzero} & \\ & 0 & \ddots & \cdot \\ & & & d_n \end{bmatrix}$$

upper triangular

triangular

Theorem (Eigenvalues of a ~~diagonal~~ matrix)

For a triangular matrix

1. *the determinant is the product of the entries on the main diagonal.*
2. *the eigenvalues are the entries on the main diagonal.*

Prove the results.

(2)

$$\left| \begin{bmatrix} d_1 & \cdot & \cdot & \cdots \\ & d_2 & & \vdots \\ & & d_3 & \\ 0 & & & \ddots \\ & & & d_n \end{bmatrix} \right| = d_1 \left| \begin{bmatrix} d_2 & & \neq 0 \\ & d_3 & \\ 0 & & d_n \end{bmatrix} \right| = d_1 \, d_2 \left| \begin{matrix} d_3 & \neq 0 \\ 0 & d_n \end{matrix} \right|$$

$$\vdots$$

$$= d_1 \cdot d_2 \cdot \ldots \cdot d_n.$$

(2) Find λ such that

$$\left| \begin{bmatrix} d_1-\lambda & & \text{nonzeros} \\ & d_2-\lambda & \\ \text{zeros} & & \ddots & \\ & & & d_n-\lambda \end{bmatrix} \right| = (d_1-\lambda)(d_2-\lambda)\cdot \ldots \cdot(d_n-\lambda) = 0$$

So eigenvalues are

$d_1, d_2, \ldots, d_n$.

The *trace* of a square matrix $\mathbf{A}$, denoted $\text{tr}(\mathbf{A})$, is the sum of its diagonal entries.

Theorem (Properties of the trace)

For any matrices $\mathbf{A}$ *and* $\mathbf{B}$*, we have*

1. $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$.
2. $\text{tr}(\mathbf{A}^T\mathbf{A}) = \sum_i \sum_j a_{ij}^2$, *where* a_{ij} *are the entries of* $\mathbf{A}$.

$\det(tI - A)$

The function $p_{\mathbf{A}}(t) =$ [redacted] is called the *characteristic polynomial* of $\mathbf{A}$.

Theorem (Expansion of characteristic polynomial)

The characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$ has the terms

$$p_{\mathbf{A}}(t) = t^n - (\operatorname{tr}\mathbf{A})t^{n-1} + \cdots + (-1)^n \det \mathbf{A}.$$

Exercise: For an $n \times n$ matrix $\mathbf{A}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, use above to show

1. $\operatorname{tr}\mathbf{A} = \sum_{i=1}^n \lambda_i$
2. $\det \mathbf{A} = \prod_{i=1}^n \lambda_i$

If $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A, then we can write

$$p_A(t) = (t-\lambda_1)(t-\lambda_2)\cdot \ldots \cdot (t-\lambda_n) = t^n - (\lambda_1 + \lambda_2 + \ldots + \lambda_n)t^{n-1} + \ldots + (-1)^n \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$$

Theorem (Further properties of the determinant)

1. $|\mathbf{A}^{-1}| = 1/|\mathbf{A}|$
2. $|c\mathbf{A}| = c^n|\mathbf{A}|$ *if* $\mathbf{A}$ *is* $n \times n$
3. $\begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{vmatrix} = |\mathbf{A}||\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}| = |\mathbf{D}||\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}|.$

See Res A.18 of Monahan (2008).

$$\underset{n \times n}{A} = P \underset{\text{diagonal}}{D} P^{-1}$$

A square matrix $\mathbf{A}$ is *diagonalizable* if $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ with $\mathbf{P}$ invertible, $\mathbf{D}$ diagonal.

Theorem (Sufficient and necessary conditions for diagonalizability)

[redacted]

P is invertible

[redacted] $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ *with* $\mathbf{D}$ *diagonal iff*

1. *the columns of* $\mathbf{P}$ *are n linearly independent eigenvectors of* $\mathbf{A}$*, and*
2. *the diagonal entries of* $\mathbf{D}$ *are the corresponding eigenvalues of* $\mathbf{A}$.

Prove the result.

"$\Leftarrow$" Assume ① and ②. Then

$$AP = A[\underbrace{v_1 \cdots v_n}_{\text{linearly indep. eigenvectors}}]$$

$$= [A\underset{\sim}{v}_1 \cdots A\underset{\sim}{v}_n]$$

$$= [\lambda_1 \underset{\sim}{v}_1 \quad \cdots \quad \lambda_n \underset{\sim}{v}_n]$$

$$= [\underset{\sim}{v}_1 \cdots \underset{\sim}{v}_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$= \underset{n\times n}{P} \; D$$

Since $\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_n$ are linearly indep., P is invertible.

Therefore we have

$$A \underbrace{P P^{-1}}_{I} = P D P^{-1}$$

$$\Rightarrow \quad A = P D P^{-1}.$$

"$\Rightarrow$" Assume $A = P D P^{-1}$, D diag, P invertible

Then $AP = PD$,

where

$$AP = A[\underset{\sim}{p}_1, \dots, \underset{\sim}{p}_n]$$

$$= [A\underset{\sim}{p}_1 \cdots A\underset{\sim}{p}_n]$$

and $$PD = [\underset{\sim}{p}_1 \cdots p_n] \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} = [d_1 \underset{\sim}{p}_1 \cdots d_n \underset{\sim}{p}_n]$$

$\Rightarrow \quad [\,A\underset{\sim}{p}_1 \;\cdots\; A\underset{\sim}{p}_n\,] = [\,d_1\underset{\sim}{p}_1 \;\cdots\; d_n\underset{\sim}{p}_n\,]$

$\Rightarrow \quad A\underset{\sim}{p}_j = d_j\underset{\sim}{p}_j \quad$ for $\quad j = 1, \dots, n$,

so $\underset{\sim}{p}_1, \dots, \underset{\sim}{p}_n$ must be eigenvectors corresponding to the eigenvalues $d_1, \dots, d_n$.

Moreover, $\underset{\sim}{p}_1, \dots, \underset{\sim}{p}_n$ are linearly indep. since P is invertible.

$A = PDP^{-1}$ Symmetric: $A = A^T$

Exercise: If possible, diagonalize the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Steps:

1. Find the eigenvalues of $\mathbf{A}$. $\lambda_1 = 1$ $\lambda_2 = -2$, $\lambda_3 = -2$.
2. Find three linearly indep. eigenvectors (if not possible, $\mathbf{A}$ not diagonalizable).
3. Give (if possible) the diagonalization $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$.

(2) Find eigenvector corresponding to $\lambda_1 = 1$.

$A\underset{\sim}{x} = \lambda_1 \underset{\sim}{x} \iff (A - \lambda_1 I)\underset{\sim}{x} = \underset{\sim}{0}$

$$[A - \lambda_1 I \quad \underset{\sim}{0}] \sim \begin{bmatrix} 1-1 & 3 & 3 & 0 \\ -3 & -5-1 & -3 & 0 \\ 3 & 3 & 1-1 & 0 \end{bmatrix}$$

$$\vdots$$

$$\sim \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\underset{\sim}{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Find eigenvectors corresponding to $\lambda_2 = -2$, $\lambda_3 = -2$.

$$\left[A - (-2)I \quad \underset{\sim}{0} \right] = \begin{bmatrix} 1+2 & 3 & 3 & 0 \\ -3 & -5+2 & -3 & 0 \\ 3 & 3 & 1+2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\underset{\sim}{v}_2} + x_3 \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\underset{\sim}{v}_3}$$

$$\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\} = \left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{\lambda = 1}, \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\lambda = -2} \right\}$$

linearly independent.

$$A = PDP^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}$$

Lay, D. C. (2003). *Linear algebra and its applications. Third edition.* Pearson Education.

Monahan, J. F. (2008). *A primer on linear models.* CRC Press.