STAT 824 sp 2023 Lec 09 slides

Edgeworth expansion and second-order correctness of the bootstrap

$X_1, \ldots, X_n \sim F$

$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{D} N(0,1)$

$T^*_{n,U} = \sqrt{n}(\bar{X}^*_n - \bar{X}_n)$

$T_{n,U} = \sqrt{n}(\bar{X}_n - \mu)$

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$T^*_{n,S} = \frac{\sqrt{n}(\bar{X}^*_n - \bar{X}_n)}{\hat{\sigma}^*_n}$

$T_{n,S} = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\hat{\sigma}_n}$

Better

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

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"Landau" notation

Asymptotic notation

Let a_n and b_n be sequences of constants with $a_n > 0$ and let U_n be a seq. of rvs.

- Little "o":
  1. If $b_n/a_n \to 0$ as $n \to \infty$ we write $b_n = o(a_n)$. b_n is of smaller order than a_n
  2. If $U_n/a_n \xrightarrow{p} 0$ as $n \to \infty$ we write $U_n = o_p(a_n)$. U_n is of smaller order (probabilistically) than a_n.
- Big "O": $\frac{b_n}{a_n} < C$
  1. If $b_n \leq Ca_n$ for all n we write $b_n = O(a_n)$. b_n is of order a_n (or smaller)
  2. If $\forall \varepsilon > 0 \; \exists C$ such that $P(|U_n| > Ca_n) < \varepsilon$ for all n we write $U_n = O_p(a_n)$.

     $P(|U_n| \leq Ca_n) > 1 - \varepsilon$

$\frac{U_n}{a_n}$ tends not to get to big.

If V_n is also a sequence of rvs:

1. We write $U_n = o_p(V_n)$ if $U_n/V_n = o_p(1)$. $U_n/V_n \xrightarrow{p} 0.$
2. We write $U_n = O_p(V_n)$ if $\boxed{U_n/V_n = O_p(1).}$

"Bounded in probability"

$P(|U_n/V_n| > C) < \varepsilon \quad \forall n.$

$P(|U_n/V_n| \leq C) > 1 - \varepsilon$

$\frac{\bar{Z}_n}{1} \xrightarrow{p} 0$

$\bar{Z}_n \xrightarrow{p} 0$

For $Z_1, \ldots, Z_n$ ind with mean zero and finite variance, $\bar{Z}_n = n^{-1}(Z_1 + \cdots + Z_n)$:

$$\bar{Z}_n = o_p(1), \quad \underbrace{\sqrt{n}\bar{Z}_n}_{\to N(0,v)} = O_p(1), \quad \bar{Z}_n = O_p(n^{-1/2}).$$

For $U_1, \ldots, U_n \overset{\text{ind}}{\sim}$ Uniform$(0,1)$, $U_{(1)} \leq \cdots \leq U_{(n)}$ the order statistics:

$$\boxed{U_{(1)} = o_p(1),} \quad \boxed{U_{(1)} = o_p(n^{-1/2}),} \quad U_{(1)} = O_p(n^{-1}).$$

equiv.

$U_{(1)} \xrightarrow{p} 0$

$\frac{U_{(1)}}{n^{-1/2}} = \sqrt{n}\, U_{(1)} \xrightarrow{p} 0.$

$n U_{(1)} = O_p(1)$

Recall the central limit theorem:

For $X_1, \dots, X_n$ iid with mean μ and variance $\sigma^2 < \infty$,

$$\sqrt{n}(\bar{X}_n - \mu)/\sigma \to \text{Normal}(0, 1)$$

in distribution as $n \to \infty$.

But we might want to know:

- How fast is the convergence?
- What features of the distribution of $X_1, \dots, X_n$ affect the rate and how?

Edgeworth expansions help us answer these questions.

$$\frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\sigma} \xrightarrow{D} N(0,1) \quad \text{as} \quad n \to \infty.$$

that is

$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\sigma} \le x \right) - \Phi(x) \right| \to 0 \quad \text{as} \quad n \to \infty.$$

First-order EE:

$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\sigma} \le x \right) - \left\{ \Phi(x) - \frac{1}{6\sqrt{n}} \frac{\mu_3}{\sigma^3} (x^2 - 1)\, \phi(x) \right\} \right| = o(n^{-1/2})$$

$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\sigma} \le x \right) - \left\{ \begin{aligned} &\Phi(x) - \frac{1}{6\sqrt{n}} \frac{\mu_3}{\sigma^3} (x^2 - 1)\, \phi(x) \\ &+ \frac{1}{24} \frac{1}{n} \left(\frac{\mu_4}{\sigma^4} - 3 \right)(x^3 - 3x)\, \phi(x) \\ &+ \frac{1}{72n} \frac{\mu_3^2}{\sigma^6} (x^5 - 10x^3 + 15x)\, \phi(x) \end{aligned} \right\} \right| = o(n^{-1})$$

1st- and 2nd-order Edgeworth expansions. See Peter Hall's book [2].

Let $Y_1, \dots, Y_n$ be iid, $\mathbb{E}Y_1 = \mu$, $\operatorname{Var} Y_1 = \sigma^2 \in (0, \infty)$, $\mathbb{E}|Y_1|^3 < \infty$, $\mathbb{E}|Y_1|^4 < \infty$.

Also, suppose $\lim_{|t|\to\infty} |\mathbb{E}\exp(\iota t Y_1)| < 1$ (Cramer's condition). Ignore for now

Then

cdf of $\sqrt{n}(\bar{Y}_n-\mu)/\sigma$

$$\sqrt{n}\cdot\sup_{x\in\mathbb{R}} \left| \underline{P(\sqrt{n}(\bar{Y}_n - \mu)/\sigma \le x)} - \boxed{\Psi_{n,3}(x)} \right| \longrightarrow 0$$

$$n\cdot\sup_{x\in\mathbb{R}} \left| P(\sqrt{n}(\bar{Y}_n - \mu)/\sigma \le x) - \underline{\Psi_{n,4}(x)} \right| \longrightarrow 0$$

as $n \to \infty$, where

γ skewness $\leftarrow H_2(x)$

diff in kurtosis from $N(0,1)$ $\leftarrow H_3(x)$

τ γ^2 skewness

$$\Psi_{n,3}(x) = \Phi(x) - \frac{1}{6\sqrt{n}}\boxed{\frac{\mu_3}{\sigma^3}}(x^2 - 1)\phi(x)$$

$$\Psi_{n,4}(x) = \Psi_{n,3}(x) - \left[\frac{1}{24n}\left(\boxed{\frac{\mu_4}{\sigma^4}} - 3\right)(x^3 - 3x) + \frac{1}{72n}\boxed{\frac{\mu_3^2}{\sigma^6}}(x^5 - 10x^3 + 15x)\right]\phi(x),$$

Hermite polynomials $H_5(x)$

In the above, $\mu_3 = \mathbb{E}(Y_1 - \mu)^3$ and $\mu_4 = \mathbb{E}(Y_1 - \mu)^4$.

Discuss: The role of moments in the Edgeworth expansions.

We can get the 1st-order Edgeworth expansion under weaker assumptions:

1st-order Edgeworth expansion. See pg. 365 of Athreya and Lahiri [1].

Let $Y_1, \dots, Y_n$ be iid, $\mathbb{E} Y_1 = \mu$, Var $Y_1 = \sigma^2 \in (0, \infty)$, $\mathbb{E}|Y_1|^3 < \infty$.

Also, suppose $|\mathbb{E}\exp(\iota t Y_1)| < 1$ for all $t \neq 0$ (non-lattice).

Then

$$\sup_{x \in \mathbb{R}} |P(\sqrt{n}(\bar{Y}_n - \mu)/\sigma \leq x) - \Psi_{n,3}(x)| = o(n^{-1/2})$$

as $n \to \infty$, where

$$\Psi_{n,3}(x) = \Phi(x) - \frac{1}{6\sqrt{n}} \frac{\mu_3}{\sigma^3}(x^2 - 1)\phi(x).$$

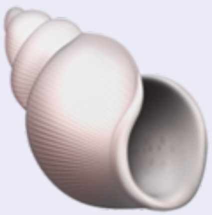
To derive the Edgeworth expansions, we will need several tools, starting with...

$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

Hermite polynomials

The *Hermite polynomials* $H_1, H_2, \dots$ are defined by the relation

$$(-1)^k \frac{d^k}{dx^k}\phi(x) = H_k(x)\phi(x), \quad k = 1, 2, \dots$$



Exercise: Find the first 3 Hermite polynomials.

$H_1(x) = x$

$$(-1)\frac{d}{dx}\phi(x) = (-1)\frac{d}{dx}\frac{1}{\sqrt{2\pi}}e^{-x^2/2} = (-1)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}(-x) = x\,\phi(x) = H_1(x)\,\phi(x)$$

$$(-1)^2 \frac{d^2}{dx^2}\phi(x) = \frac{d}{dx}\frac{1}{\sqrt{2\pi}} e^{-x^2/2}(-x)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2}(-x)(-x) - 1 \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$= (x^2-1)\,\phi(x)$$

$$H_2(x) = x^2 - 1 .$$

$$= H_2(x)\,\phi(x)$$

$\vdots$

$M_X(t) = \mathbb{E}e^{tX}$ $\psi_X(t) = \mathbb{E}e^{\imath tX}$

characteristic function $\imath = \sqrt{-1}$

Inversion formula

If X is a rv with ch. function ψ_X such that $\int_{-\infty}^{\infty} |\psi_X(t)| dt < \infty$, then X has pdf

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\iota tx)\psi_X(t)dt \quad \text{for all } x \in \mathbb{R}.$$

$M_Z(t) = e^{t^2/2}$, $Z \sim N(0,1)$

Exercise: Use the inversion formula to establish the useful identity

$$\boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\iota tx} e^{-t^2/2} (\iota t)^k dt = H_k(x)\phi(x).}$$

Exercise: Let $X_1, \ldots, X_n$ be iid with $P(\sqrt{n}\bar{x}_n \le x) = \Phi(x) + \boxed{\text{things}} + o(n^{-1})$.

$$\mathbb{E}X_1 = 0, \quad \mathbb{E}X_1^2 = 1, \quad \mathbb{E}X_1^3 = \gamma, \quad \mathbb{E}X_1^4 = \tau$$

and $|\psi_X(t)| < 1$ for all $t \neq 0$, where $\psi_X(t) = \mathbb{E}\exp(\iota t X_1)$.

$\sqrt{n}\bar{x}_n = \frac{\sqrt{n}(\bar{x}_n - \mu)}{\sigma}$

Derive the Edgeworth expansion for $\sqrt{n}\bar{X}_n$ in these steps:

1. Write the characteristic function of $S_n = \sqrt{n}\bar{X}_n$ as $[\psi_X(t/\sqrt{n})]^n$.
2. Taylor expand $\psi_X(t/\sqrt{n})$ around $t = 0$.
3. Raise expansion to power n, discarding terms of order $o(n^{-1})$.
4. Make use of this fact: For each nonnegative integer k,

$$\left(1 + \frac{a}{n}\right)^{n-k} = e^a\left[1 - \frac{a(a+k)}{2n}\right] + o(n^{-1}) \quad \text{as } n \to \infty.$$

5. Again discard terms of order $o(n^{-1})$ to get approximation $\tilde{\psi}_{S_n}$ to ψ_{S_n}.
6. Use inversion formula to invert $\tilde{\psi}_{S_n}$ into the corresponding pdf $\tilde{f}_{S_n}$.
7. Take the antiderivative of $\tilde{f}_{S_n}$ using $\frac{d}{dx}[H_k(x)\phi(x)] = -H_{k+1}(x)\phi(x)$.

$$P\left(\frac{\sqrt{n}\,(\bar{Y}_n-\mu)}{\sigma}\le x\right) = \underbrace{\Phi(x) + \boxed{}}_{\Psi_{n,4}} + \underbrace{o(n^{-1})}$$

something of smaller order than n^{-1}.

$$S_n = \sqrt{n}\,\bar{X}_n$$

$$\begin{aligned}
\psi_{S_n}(t) &= \mathbb{E}\, e^{itS_n} \\
&= \mathbb{E}\, e^{it\sqrt{n}\,\bar{X}_n} \\
&= \mathbb{E}\, e^{it\sqrt{n}\,\frac{1}{n}(X_1+\dots+X_n)} \\
&= \mathbb{E}\, e^{it/\sqrt{n}\,(X_1+\dots+X_n)} \\
&= \mathbb{E}\prod_{i=1}^{n} e^{it/\sqrt{n}\,X_i} \\
&= \prod_{i=1}^{n} \mathbb{E}\, e^{it/\sqrt{n}\,X_i} \\
&= \prod_{i=1}^{n} \psi_{X_i}(t/\sqrt{n}) \\
&= \left[\psi_{X_1}(t/\sqrt{n})\right]^n \\
&= \left[\mathbb{E}\, e^{it/\sqrt{n}\,X_1}\right]^n \\
&= \left[\mathbb{E}\left(1 + \frac{it}{\sqrt{n}}X_1 + \frac{(it)^2}{2n}X_1^2 + \frac{(it)^3}{3!\,n^{3/2}}X_1^3 + \frac{(it)^4 X_1^4}{4!\,n^2} + o(n^{-2})\right)\right]^n \\
&= \left[\left(1 + \frac{(it)^2}{2n}\right) + \frac{(it)^3\gamma}{6\,n^{3/2}} + \frac{(it)^4\tau}{24\,n^2} + o(n^{-2})\right]^n
\end{aligned}$$

$$X \perp\!\!\!\perp Y$$

$$\mathbb{E}XY = \mathbb{E}X\,\mathbb{E}Y$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\left[\left(0^{th} \text{ order Edgeworth expansion } (\text{keep only 1st term}) \right) \approx \left(\left[1 - \frac{t^2}{2n} \right]^n \to e^{-t^2/2} . \right) \right]$$

$$(a_1 + \dots + a_m)^n = \sum_{\substack{n_1, \dots, n_m \in \{0, \dots, n\} \\ n_1 + \dots + n_m = n}} \left(\frac{n!}{n_1! \cdot \ldots \cdot n_m!} \right) a_1^{n_1} \cdot \ldots \cdot a_m^{n_m}$$

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x} = \sum_{x=0}^{n} \frac{n!}{x!(n-x)!} a^x b^{(n-x)}$$

$$= \left(1 - \frac{t^2/2}{n} \right)^n + \frac{n!}{(n-1)!1!} \left(1 - \frac{t^2/2}{n} \right)^{n-1} \frac{(it)^3 \gamma}{6 n^{3/2}}$$

$$+ \frac{n!}{(n-1)!} \left(1 - \frac{t^2/2}{n} \right)^{n-1} \frac{(it)^4 \tau}{24 n^2}$$

$$+ \frac{n!}{(n-2)!\,2!} \left(1 - \frac{t^2/2}{n} \right)^{n-2} \left(\frac{(it)^3 \gamma}{6 n^{3/2}} \right)^2 + o(n^{-2})$$

$$= \left(1 - \frac{t^2/2}{n} \right)^n + \left(1 - \frac{t^2/2}{n} \right)^{n-1} \frac{(it)^3 \gamma}{6 n^{1/2}}$$

$$+ \left(1 - \frac{t^2/2}{n} \right)^{n-1} \frac{(it)^4 \tau}{24 n}$$

$$+ \frac{(n-1)}{n^2} \left(1 - \frac{t^2/2}{n} \right)^{n-2} \frac{(it)^6 \gamma^2}{72} + o(n^{-2})$$

$\vdots$

$$\underset{\substack{\uparrow \\ \sqrt{n}\bar{x}_n}}{\psi_{S_n}(t)} = e^{-t^2/2} \left[1 + \frac{\gamma (it)^3}{6\sqrt{n}} + \frac{(\tau - 3)(it)^4}{24 n} + \frac{\gamma^2 (it)^6}{72 n} \right] + o(n^{-1})$$

$$\boxed{\tilde{\psi}_{S_n}(t) = e^{-t^2/2} \left[1 + \frac{\gamma (it)^3}{6\sqrt{n}} + \frac{(\tau - 3)(it)^4}{24 n} + \frac{\gamma^2 (it)^6}{72 n} \right]}$$

Use inversion formula

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-\iota t x)\psi_X(t)dt \quad \text{for all } x \in \mathbb{R}.$$

$$\tilde{f}_{S_n}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\iota t x} e^{-t^2/2}\left[1 + \frac{\gamma(\iota t)^3}{6\sqrt{n}} + \frac{(\tau-3)(\iota t)^4}{24n} + \frac{\gamma^2(\iota t)^6}{72n}\right]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\iota t x} e^{-t^2/2}(\iota t)^0 dt = \phi(x)$$

$$+ \frac{\gamma}{6\sqrt{n}} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\iota t x} e^{-t^2/2}(\iota t)^3 dt = H_3(x)\phi(x)$$

$$+ \frac{(\tau-3)}{24n} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\iota t x} e^{-t^2/2}(\iota t)^4 dt = H_4(x)\phi(x)$$

$$+ \frac{\gamma^2}{72n} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\iota t x} e^{-t^2/2}(\iota t)^6 dt = H_6(x)\phi(x)$$

Density of $\sqrt{n}\bar{X}_n$ is approximated by

$$\tilde{f}_{S_n}(x) = \phi(x) + \frac{\gamma}{6\sqrt{n}} H_3(x)\phi(x) + \frac{(\tau-3)}{24n} H_4(x)\phi(x) + \frac{\gamma^2}{72n} H_6(x)\phi(x).$$

$$\frac{d}{dx}\left[H_k(x)\phi(x)\right] = -H_{k+1}(x)\phi(x), \quad \text{i.e.} \quad \int H_k(x)\phi(x) = -H_{k-1}(x)\phi(x)$$

$$\tilde{F}_{S_n}(x) = \Phi(x) - \frac{\gamma}{6\sqrt{n}} H_2(x)\phi(x) - \frac{(\tau-3)}{24n} H_3(x)\phi(x) - \frac{\gamma^2}{72n} H_5(x)\phi(x).$$

$$= \Phi - \phi(x)\left[\frac{\gamma}{6\sqrt{n}}\underbrace{H_2(x)}_{x^2-1} + \frac{(\kappa-3)}{24n}\underbrace{H_3(x)}_{x^3-3x} + \frac{\gamma^2}{72n}\underbrace{H_5(x)}_{x^5-10x^3+15x}\right]$$

<u>**Example:**</u> Let $Y_1, \dots, Y_n \overset{ind}{\sim}$ Exponential (λ).

Let $X_i = \dfrac{Y_i - \lambda}{\lambda}$, study $\sqrt{n}\,\bar{X}_n \xrightarrow{D} N(0,1)$.

$$\sqrt{n}\,\bar{X}_n = \sqrt{n}(\bar{X}_n - \mu)/\sigma \xrightarrow{D} N(0,1).$$

$\mathbb{E}X_1 = 0$ $\qquad \mathbb{E}X_1^3 = \mathbb{E}\left(\dfrac{Y_1-\lambda}{\lambda}\right)^3 = \cdots = 2.$ $\quad \mathbb{E}X_1^4 = 9.$

$\mathrm{Var}\,X_1 = \mathrm{Var}\left(\dfrac{Y_1-\lambda}{\lambda}\right) = \dfrac{1}{\lambda^2}\mathrm{Var}\,Y_1 = \dfrac{\lambda^2}{\lambda^2} = 2$

① Find pdf of $\sqrt{n}\,\bar{X}_n$.

Write $\sqrt{n}\,\bar{X}_n = \sqrt{n}\left(\dfrac{1}{n}\sum_{i=1}^{n}\dfrac{Y_i-\lambda}{\lambda}\right) = \sqrt{n}\,\dfrac{(\bar{Y}_n-\lambda)}{\lambda} = \dfrac{\sqrt{n}\,\bar{Y}_n}{\lambda} - \sqrt{n}.$

Then
$$M_{\sqrt{n}\bar{Y}_n}(t) = M_{\frac{1}{\sqrt{n}}(Y_1+\dots+Y_n)}(t)$$
$$= M_{Y_1+\dots+Y_n}(t/\sqrt{n})$$
$$= \left(M_{Y_1}(t/\sqrt{n})\right)^n$$
$$= \left[\left(1-\lambda(t/\sqrt{n})\right)^{-1}\right]^n$$
$$= \left(1-\frac{\lambda}{\sqrt{n}}t\right)^{-n}, \text{ mgf of Gamma}\left(n, \frac{\lambda}{\sqrt{n}}\right).$$

Let $S = \sqrt{n}\,\bar{Y}_n$. Then

$$f_S(s) = \frac{1}{\Gamma(n)\left(\frac{\lambda}{\sqrt{n}}\right)^n} s^{n-1} e^{-s/\lambda/\sqrt{n}} \mathbb{1}(s>0)$$

Now do transformation:

$$\underline{(\sqrt{n}\bar{X}_n)} = V = \frac{S}{\lambda} - \sqrt{n} = g(s) \quad \Leftrightarrow \quad S = (V+\sqrt{n})\lambda = g^{-1}(v)$$

$$\frac{d}{dv} g^{-1}(v) = \lambda$$

$$\sqrt{n}\bar{X}_n \sim f_V(v) = \frac{1}{\Gamma(n)\left(\frac{\lambda}{\sqrt{n}}\right)^n} [(v+\sqrt{n})\lambda]^{n-1} e^{-\frac{(v+\sqrt{n})\lambda}{\lambda/\sqrt{n}}} \lambda \, \mathbb{1}(v>-\sqrt{n})$$

$$= \frac{1}{\Gamma(n)} \frac{1}{1/\sqrt{n}} \left(\frac{v+\sqrt{n}}{1/\sqrt{n}}\right)^{n-1} e^{-\left(\frac{v+\sqrt{n}}{1/\sqrt{n}}\right)} \mathbb{1}(v>-\sqrt{n}).$$

$$\longrightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}$$

1st- and 2nd-order Edgeworth expansions for studentized pivot

Let $Y_1, \dots, Y_n$ be iid, $\mathbb{E} Y_1 = \mu$, Var $Y_1 = \sigma^2 \in (0, \infty)$, $\mathbb{E}|Y_1|^3 < \infty$, $\mathbb{E}|Y_1|^4 < \infty$.

Also, suppose $\lim_{|t| \to \infty} |\mathbb{E} \exp(\iota t Y_1)| < 1$ (Cramer's condition).

Then

$$\sup_{x \in \mathbb{R}} |P(\sqrt{n}(\bar{Y}_n - \mu)/\hat{\sigma}_n \leq x) - \tilde{\Psi}_{n,3}(x)| = o(n^{-1/2})$$

$$\sup_{x \in \mathbb{R}} |P(\sqrt{n}(\bar{Y}_n - \mu)/\hat{\sigma}_n \leq x) - \tilde{\Psi}_{n,4}(x)| = o(n^{-1})$$

as $n \to \infty$, where

$$\tilde{\Psi}_{n,3}(x) = \Phi(x) + \frac{1}{6\sqrt{n}} \frac{\mu_3}{\sigma^3}(2x^2 + 1)\phi(x)$$

$$\tilde{\Psi}_{n,4}(x) = \tilde{\Psi}_{n,3}(x) + \left[\frac{1}{12n}(\frac{\mu_4}{\sigma^4} - 3)(x^3 - 3x) - \frac{1}{18n}\frac{\mu_3^2}{\sigma^6}(x^5 + 2x^3 - 3x) - \frac{1}{4n}(x^3 + 3x)\right]\phi(x).$$

Remember our pivot quantities for the mean:

$$T_{n,\mathrm{U}} \cancel{=} \boxed{\sqrt{n}(\bar{X}_n - \mu)}, \quad T_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}, \quad T_{n,\mathrm{S}} = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\hat{\sigma}_n}.$$

Accuracies of Normal approximations to pivot distributions

Letting $G_{n,U}$, G_n, and $G_{n,S}$ be the cdfs of these pivots, we obtain

$$\sup_{x\in\mathbb{R}} |G_{n,U}(x) - \Phi(x/\sigma)| \cancel{=} O(n^{-1/2})$$

$$\sup_{x\in\mathbb{R}} |G_n(x) - \Phi(x)| = O(n^{-1/2})$$

$$\sup_{x\in\mathbb{R}} |G_{n,S}(x) - \Phi(x)| = O(n^{-1/2}).$$

as $n \to \infty$ from the Edgeworth expansions.

So the error of the Normal approximation to these cdfs is of the order $O(n^{-1/2})$.

Edgeworth expansion results for the bootstrap

Let $X_1, \dots, X_n$ be iid, $\mathbb{E}X_1 = \mu$, $\operatorname{Var} X_1 = \sigma^2 \in (0, \infty)$, $\mathbb{E}|X_1|^3 < \infty$.

Also, suppose $|\mathbb{E}\exp(\iota t X_1)| < 1$ for all $t \neq 0$ (non-lattice). Then

bootstrap — true cdf of $\sqrt{n}(\bar{X}_n - \mu)$

$\sqrt{n}(\bar{X}_n^* - \bar{X}_n)$

$$\sup_{x\in\mathbb{R}} |\boxed{\hat{G}_{n,U}(x)} - \boxed{G_{n,U}(x)}| = O_p(n^{-1/2})$$

true cdf of $\sqrt{n}(\bar{X}_n - \mu)/\sigma$

$$\sup_{x\in\mathbb{R}} |\hat{G}_n(x) - G_n(x)| = o_p(n^{-1/2})$$

$$\boxed{\sup_{x\in\mathbb{R}} |\boxed{\hat{G}_{n,S}(x)} - \boxed{G_{n,S}(x)}| = o_p(n^{-1/2}).}$$

$\boxed{\sqrt{n}(\bar{X}_n^* - \bar{X}_n)/\hat{\sigma}_n^*}$ — true cdf of $\dfrac{\sqrt{n}(\bar{X}_n - \mu)}{\hat{\sigma}_n}$

as $n \to \infty$.

Note that the accuracy $\boxed{o_p(n^{1/2})}$ is better than $\boxed{O_p(n^{-1/2}) \text{ and } O(n^{-1/2}).}$

The bootstrap estimators $\hat{G}_n(x)$ and $\hat{G}_{n,S}(x)$ are *second-order correct*.

Exercise: Sketch the proof of each of the results above.

How to show?

$$\sup_{x\in\mathbb{R}} \left| \underbrace{\hat{G}_{n,s}(x)}_{\hat{a}} - \underbrace{G_{n,s}(x)}_{a} \right| = o_p(n^{-1/2})$$

(i) $$\sup_{x\in\mathbb{R}} \left| P\left(\frac{\sqrt{n}(\bar{x}_n - \mu)}{\hat{\sigma}_n} \le x \right) - \left\{ \Phi(x) + \frac{1}{6\sqrt{n}} \frac{\mu_3}{\sigma^3} (2x^2+1) \phi(x) \right\} \right| = o(n^{-1})$$

(ii) $$\sup_{x\in\mathbb{R}} \left| P_*\left(\frac{\sqrt{n}(\bar{x}^* - \bar{x}_n)}{\hat{\sigma}_n^*} \le x \right) - \left\{ \Phi(x) + \frac{1}{6\sqrt{n}} \frac{\hat{\mu}_3}{\hat{\sigma}^3} (2x^2+1) \phi(x) \right\} \right| = o_p(n^{-1})$$

$$|\hat{a} - a| = |(\hat{a} - \hat{b}) - (a-b) + (\hat{b} - b)|$$

$$\le |\hat{b} - b| + |\hat{a} - \hat{b}| + |a-b|$$

Lattice random variables

A rv X is *lattice* if there exist an $a \in \mathbb{R}$ and $h \neq 0$ s.t.

$$P(X \in \{a + jh : j \in \mathbb{Z}\}) = 1.$$

Prop 10.1.1 of [1] gives

$$X \text{ is lattice} \iff |\mathbb{E} \exp \iota t X| = 1 \text{ for some } t \neq 0.$$

- A rv X can be discrete but non-lattice: e.g. X with support on $\{1, e, \pi\}$.
- There are Edgeworth expansions for lattice rvs. See pg. 367 of [1].
- If we assume $\mathbb{E}|X_1|^4 < \infty$ and *Cramer's condition*,

$$\limsup_{|t| \to \infty} |\mathbb{E} \exp(\iota t X_1)| < 1,$$

  then we can show even greater gains in accuracy from the bootstrap. See [2].
- For X with differentiable cdf F, with F' bounded, $\lim_{|t| \to \infty} |\mathbb{E} \exp(\iota t X_1)| = 0$. See Prop 12.1.2 of [1] and definition of absolute continuity on pg. 128.

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