

STAT 824 sp 2023 Lec 11 slides

Bootstrap for regression

Karl B. Gregory

University of South Carolina

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

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 - Bootstrap confidence bands

1 Bootstrap in multiple linear regression

- Constant error term variance
- Heteroscedastic error term variances
- Resample pairs

2 Pointwise confidence intervals in nonparametric regression

3 Confidence bands in nonparametric regression

- Asymptotic distribution approach
- Bootstrap confidence bands

Linear regression model

Let $(\mathbf{x}_1, Y_1), \dots, (\mathbf{x}_n, Y_n)$ be data pairs such that

$$Y_i = \mathbf{x}_i^T \boldsymbol{\beta}_0 + \varepsilon_i, \quad i = 1, \dots, n,$$

with $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^p$ deterministic, $\varepsilon_1, \dots, \varepsilon_n$ iid with $\mathbb{E}\varepsilon_1 = 0$, $\mathbb{E}\varepsilon_1^2 = \sigma^2 < \infty$.

Let $\mathbf{X} = [\mathbf{x}_1, \dots, \mathbf{x}_n]^T$ and

$$\hat{\boldsymbol{\beta}}_n = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y}$$

$$\hat{\sigma}_n^2 = (n - p)^{-1} \|\mathbf{Y} - \mathbf{X} \hat{\boldsymbol{\beta}}_n\|_2^2$$

Exercise: Consider estimating contrasts $\mathbf{c}^T \boldsymbol{\beta}_0$ for $\mathbf{c} \in \mathbb{R}^p$ with $\mathbf{c}^T \hat{\boldsymbol{\beta}}_n$.

- 1 Come up with pivot quantities relevant for making inferences.
- 2 Give the form of a confidence interval for $\mathbf{c}^T \boldsymbol{\beta}_0$.

Define

$$\sigma_{\mathbf{c},n}^2 = [\mathbf{c}^T (n^{-1} \mathbf{X}^T \mathbf{X})^{-1} \mathbf{c}]^{1/2} \cdot \sigma^2$$

$$\hat{\sigma}_{\mathbf{c},n}^2 = [\mathbf{c}^T (n^{-1} \mathbf{X}^T \mathbf{X})^{-1} \mathbf{c}]^{1/2} \cdot \hat{\sigma}_n^2$$

Theorem (Asymptotically Normal pivots for least-squares coefficients)

For $\mathbf{c} \in \mathbb{R}^p$, the pivot quantities

$$T_{n,U} = \sqrt{n} \cdot \mathbf{c}^T (\hat{\beta}_n - \beta_0) \quad \text{and} \quad T_{n,S} = \sqrt{n} \cdot \mathbf{c}^T (\hat{\beta}_n - \beta_0) / \hat{\sigma}_{\mathbf{c},n}$$

are asymptotically $\text{Normal}(0, \sigma_{\mathbf{c},\infty}^2)$ and $\text{Normal}(0, 1)$ rvs, respectively, provided

$$\max_{1 \leq i \leq n} h_{ii} \rightarrow 0 \quad \text{and} \quad \sigma_{\mathbf{c},n}^2 \rightarrow \sigma_{\mathbf{c},\infty}^2 \in (0, \infty) \quad \text{as } n \rightarrow \infty,$$

where h_{ii} , $i = 1, \dots, n$ is the i th diagonal entry of $\mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T$.

Denote by $G_{n,U}$ and $G_{n,S}$ the dists of $T_{n,U}$ and $T_{n,S}$, respectively.

Residual bootstrap for linear regression

- 1 Draw $\varepsilon_1^*, \dots, \varepsilon_n^*$ with repl. from $\hat{\varepsilon}_i = Y_i - \mathbf{x}_i^T \hat{\beta}_n$, $i = 1, \dots, n$
- 2 Set $Y_i^* = \mathbf{x}_i^T \hat{\beta}_n + \varepsilon_i^*$ for $i = 1, \dots, n$.
- 3 Compute $\hat{\beta}_n^* = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y}^*$ and $(\hat{\sigma}_n^*)^2 = (n - p)^{-1} \|\mathbf{Y}^* - \mathbf{X} \hat{\beta}_n^*\|_2^2$.
- 4 Compute the bootstrap versions of the pivots $T_{n,U}$ and $T_{n,S}$ given by

$$T_{n,U}^* = \sqrt{n} \cdot \mathbf{c}^T (\hat{\beta}_n^* - \hat{\beta}_n) \quad \text{and} \quad T_{n,S}^* = \sqrt{n} \cdot \mathbf{c}^T (\hat{\beta}_n^* - \hat{\beta}_n) / \hat{\sigma}_{c,n}^*,$$

$$\text{where } (\hat{\sigma}_{c,n}^*)^2 = [\mathbf{c}^T (n^{-1} \mathbf{X}^T \mathbf{X})^{-1} \mathbf{c}]^{1/2} \cdot (\hat{\sigma}_n^*)^2.$$

Denote by $\hat{G}_{n,U}$ and $\hat{G}_{n,S}$ the dists of $T_{n,U}^*$ and $T_{n,S}^*$ conditional on the data.

Exercise: Simulate performance of residual bootstrap CIs for (i) $\mathbf{c}^T \beta_0$ and (ii) β_{0j} .

1 Bootstrap in multiple linear regression

- Constant error term variance
- **Heteroscedastic error term variances**
- Resample pairs

2 Pointwise confidence intervals in nonparametric regression

3 Confidence bands in nonparametric regression

- Asymptotic distribution approach
- Bootstrap confidence bands

Linear regression model with heteroscedasticity

Let $(\mathbf{x}_1, Y_1), \dots, (\mathbf{x}_n, Y_n)$ be data pairs such that

$$Y_i = \mathbf{x}_i^T \boldsymbol{\beta}_0 + \varepsilon_i, \quad i = 1, \dots, n,$$

with $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^p$ deterministic, $\mathbb{E}\varepsilon_i = 0$ and $\mathbb{E}\varepsilon_i^2 = \sigma_i^2 \in (0, \infty)$, $i = 1, \dots, n$.

Exercise: Give an expression for $\text{Var}(\sqrt{n} \cdot \mathbf{c}^T \hat{\boldsymbol{\beta}}_n)$ for $\mathbf{c} \in \mathbb{R}^p$.

Define

$$\vartheta_{c,n}^2 = n \cdot \mathbf{c}^T (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \cdot \text{diag}(\sigma_1^2, \dots, \sigma_n^2) \cdot \mathbf{X} (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{c}$$

$$\hat{\vartheta}_{c,n}^2 = n \cdot \mathbf{c}^T (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \cdot \text{diag}(\hat{\varepsilon}_1^2, \dots, \hat{\varepsilon}_n^2) \cdot \mathbf{X} (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{c}$$

Theorem (Asymptotically Normal pivots for LS coefs under hetsc.)

For $\mathbf{c} \in \mathbb{R}^p$, the pivot quantities

$$T_{n,U}^{\text{het}} = \sqrt{n} \cdot \mathbf{c}^T (\hat{\beta}_n - \beta_0) \quad \text{and} \quad T_{n,S}^{\text{het}} = \sqrt{n} \cdot \mathbf{c}^T (\hat{\beta}_n - \beta_0) / \hat{\vartheta}_{c,n}$$

are asymptotically $\text{Normal}(0, \vartheta_{c,\infty}^2)$ and $\text{Normal}(0, 1)$ rvs, respectively, provided

$$\max_{1 \leq i \leq n} h_{ii}^\sigma / \sigma_i^2 \rightarrow 0 \quad \text{and} \quad \vartheta_{c,n}^2 \rightarrow \vartheta_{c,\infty}^2 \in (0, \infty)$$

as $n \rightarrow \infty$, where h_{ii}^σ is i th diag. entry of $\mathbf{X} (\mathbf{X}^T \cdot \text{diag}(\sigma_1^2, \dots, \sigma_n^2) \cdot \mathbf{X})^{-1} \mathbf{X}^T$.
(some additional moment assumps. necessary for $\hat{\vartheta}_{c,n} / \vartheta_{c,n} \rightarrow^P 1$, [5])

Denote by $G_{n,U}^{\text{het}}$ and $G_{n,S}^{\text{het}}$ the distributions of $T_{n,U}^{\text{het}}$ and $T_{n,S}^{\text{het}}$, respectively.

Wild bootstrap for linear regression

- 1 Generate indep. bootstrap residuals $\varepsilon_1^{*W}, \dots, \varepsilon_n^{*W}$ satisfying $\mathbb{E}_*[\varepsilon_i^{*W}] = 0$, $\mathbb{E}_*[(\varepsilon_i^{*W})^2] = \hat{\varepsilon}_i^2$, and $\mathbb{E}_*[(\varepsilon_i^{*W})^3] = \hat{\varepsilon}_i^3$, where $\hat{\varepsilon}_i = Y_i - \mathbf{x}_i^T \hat{\boldsymbol{\beta}}_n$, $i = 1, \dots, n$.
- 2 Set $Y_i^{*W} = \mathbf{x}_i^T \hat{\boldsymbol{\beta}}_n + \varepsilon_i^*$, $i = 1, \dots, n$.
- 3 Compute $\hat{\boldsymbol{\beta}}_n^{*W} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y}^{*W}$, $\hat{\varepsilon}_i^{*W} = Y_i^{*W} - \mathbf{x}_i^T \hat{\boldsymbol{\beta}}_n^{*W}$, $i = 1, \dots, n$.
- 4 Compute the bootstrap versions of the pivots $T_{n,U}$ and $T_{n,S}$ given by

$$T_{n,U}^{\text{het}*} = \sqrt{n} \cdot \mathbf{c}^T (\hat{\boldsymbol{\beta}}_n^{*W} - \hat{\boldsymbol{\beta}}_n) \quad \text{and} \quad T_{n,S}^{\text{het}*} = \sqrt{n} \cdot \mathbf{c}^T (\hat{\boldsymbol{\beta}}_n^{*W} - \hat{\boldsymbol{\beta}}_n) / \hat{\vartheta}_{c,n}^{*W},$$

where $(\hat{\vartheta}_{c,n}^{*W})^2 = n \cdot \mathbf{c}^T (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \cdot \text{diag}((\hat{\varepsilon}_1^{*W})^2, \dots, (\hat{\varepsilon}_n^{*W})^2) \cdot \mathbf{X} (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{c}$.

Denote by $\hat{G}_{n,U}^{\text{het}}$ and $\hat{G}_{n,S}^{\text{het}}$ the dists of $T_{n,U}^{\text{het}*}$ and $T_{n,S}^{\text{het}*}$ conditional on the data.

Two ways to obtain wild bootstrap residuals [1], [2]

- 1 Mammen (1993): For $i = 1, \dots, n$, get $V_{i,1}, V_{i,2} \stackrel{\text{ind}}{\sim} \text{Normal}(0, 1)$. Then set

$$U_i = (\delta_1 + V_{i,1}/\sqrt{2})(\delta_2 + V_{i,2}/\sqrt{2}) - \delta_1\delta_2,$$

where $\delta_1 = (3/4 + \sqrt{17}/12)^{1/2}$, $\delta_2 = (3/4 - \sqrt{17}/12)^{1/2}$. Then let

$$\varepsilon_i^{*W} = \hat{\varepsilon}_i \cdot U_i.$$

- 2 Das et al. (2019): For $i = 1, \dots, n$, generate $U_i \sim \text{Beta}(1/2, 3/2)$. Then set

$$\varepsilon_i^{*W} = \hat{\varepsilon}_i \cdot 4(U_i - 1/4).$$

Exercise: Simulate performance of wild bootstrap CIs for (i) $\mathbf{c}^T \beta_0$ and (ii) β_{0j} .

Discuss: Comparison of methods in simulation.

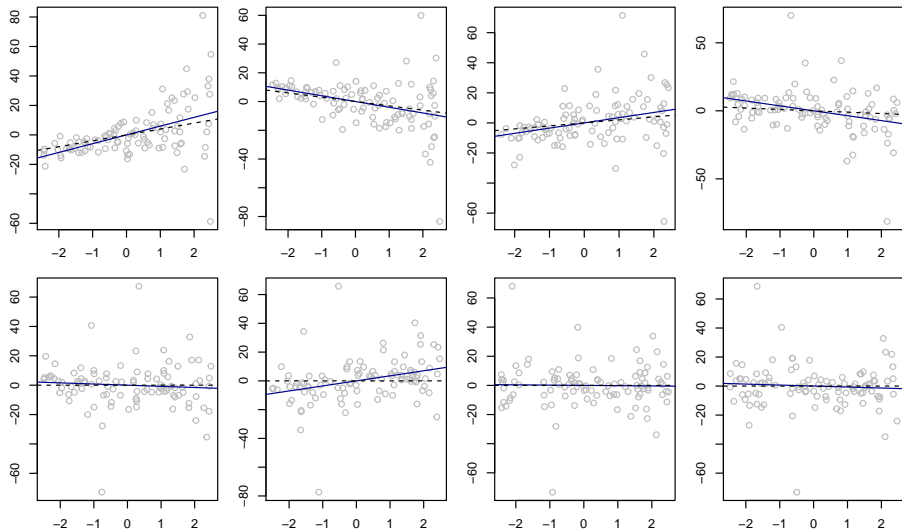
```

r <- .7
R <- r^abs( outer(1:8,1:8,"-"))
P <- 2*sin( R * pi / 6)
X <- cbind(1,(pnorm( matrix(rnorm(n*8),ncol = 8) %% chol(P)) - .5) * 5)
beta <- c(-1,c(4:1)*(-1)^(4:1),0,0,0,0)
sigma <- 1/4 + abs(X[,2] + 2.5)^2
error <- rnorm(n,0,sigma)
Y <- as.numeric(X %% beta) + error

```

Coverage of 95% confidence intervals for β_{03} at sample sizes $n = 10, 20, \dots, 100$.

method	n									
	10	20	30	40	50	60	70	80	90	100
$T_{n,U}^*$	0.34	0.81	0.88	0.91	0.93	0.95	0.93	0.95	0.95	0.95
$T_{n,S}^*$	0.98	0.99	0.98	0.97	0.97	0.97	0.97	0.98	0.97	0.96
$T_{n,S}$ as $N(0,1)$	0.66	0.92	0.94	0.96	0.95	0.97	0.96	0.97	0.96	0.95
$T_{n,U}^{\text{het}*}$	0.31	0.79	0.87	0.90	0.93	0.95	0.93	0.95	0.93	0.93
$T_{n,S}^{\text{het}*}$	0.90	0.91	0.92	0.93	0.93	0.95	0.94	0.94	0.94	0.93
$T_{n,S}^{\text{het}}$ as $N(0,1)$	0.30	0.79	0.86	0.90	0.93	0.94	0.93	0.93	0.93	0.93



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Linear regression model with heteroscedasticity and a random design

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be data pairs such that

$$Y_i = X_i^T \beta_0 + \varepsilon_i, \quad i = 1, \dots, n,$$

with $X_1, \dots, X_n \in \mathbb{R}^p$ rvs, $\mathbb{E}[\varepsilon_i | X_i] = 0$ and $\mathbb{E}[\varepsilon_i^2 | X_i] = \sigma_i^2 \in (0, \infty)$, $i = 1, \dots, n$.

A random design is often more realistic (but does it really matter?).

Mammen (1993) showed that the wild bootstrap works in the above setting.

The resampling pairs bootstrap

- 1 Draw $(X_1^*, Y_1^*), \dots, (X_n^*, Y_n^*)$ with replacement from $(X_1, Y_1), \dots, (X_n, Y_n)$.
- 2 Then let $\hat{\beta}_n^* = (\mathbf{X}^{*T} \mathbf{X}^*)^{-1} \mathbf{X}^{*T} \mathbf{Y}^*$.

Note that we must compute the inverse $(\mathbf{X}^{*T} \mathbf{X}^*)^{-1}$ for every bootstrap resample!

This is taken from Mammen (1993). “Bootstrap” is the resampling pairs bootstrap.

TABLE 1

Rates of convergence of the bootstrap procedures and the mean zero normal approximation under the assumption $E(\varepsilon_i|X_i) = 0$

Estimation of	$\mathcal{L}(\sqrt{n} \mathbf{c}^T(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}))$	$\mathcal{L}(\sqrt{n} \mathbf{c}^T(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) / \hat{\sigma}_c)$
Normal approximation $N(0, \hat{\sigma}_c^2)$	$O_p(n^{-1/2} + pn^{-1})$	
Wild bootstrap	$O_p(n^{-1/2} + pn^{-1})$	$O_p(n^{-1} + pn^{-3/2})$
Bootstrap	$O_p(n^{-1/2} + pn^{-1})$	$O_p(pn^{-1})$

In this paper the affect of the dimension p is tracked along with that of n .

The studentized resampling pairs bootstrap is more adversely affected by high dimension than the wild bootstrap.

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Nonparametric regression model

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be data pairs such that

$$Y_i = m(X_i) + \varepsilon_i, \quad i = 1, \dots, n,$$

with $X_1, \dots, X_n \in [0, 1]$ deterministic, $\mathbb{E}\varepsilon_i = 0$, $\mathbb{E}\varepsilon_i^2 = \sigma_i^2 \in (0, \infty)$, $i = 1, \dots, n$.

Consider *linear estimators*, i.e. estimators of the form

$$\hat{m}_n(x) = \sum_{i=1}^n W_{ni}(x) Y_i, \quad \text{for } x \in [0, 1].$$

Exercise: Discuss estimators of $\text{Var } \hat{m}_n(x)$ in the cases

- 1 $\sigma_1^2 = \dots = \sigma_n^2 = \sigma^2 \in (0, \infty)$.
- 2 $\sigma_1^2, \dots, \sigma_n^2$ are heteroscedastic.

Suppose we wish to build a confidence interval for $m(x)$ at some $x \in [0, 1]$.

Consider our discussions from Lecture 4: We have

$$\frac{\hat{m}_n(x) - m_n(x)}{\sqrt{\text{Var } \hat{m}_n(x)}} = \underbrace{\frac{\hat{m}_n(x) - \mathbb{E}\hat{m}_n(x)}{\sqrt{\text{Var } \hat{m}_n(x)}}}_{\rightarrow^D N(0,1)} + \underbrace{\frac{\mathbb{E}\hat{m}_n(x) - m_n(x)}{\sqrt{\text{Var } \hat{m}_n(x)}}}_{\rightarrow^P 0 \text{ if } \hat{m}_n \text{ undersmoothed}}.$$

Strategy: Undersmooth and pretend $\mathbb{E}\hat{m}_n(x)$ is equal to $m(x)$.

Studentized pivots under constant and heteroscedastic variances

$$T_{n,x} = \frac{\hat{m}_n(x) - \mathbb{E}\hat{m}_n(x)}{\hat{\sigma}_n \sqrt{\sum_{i=1}^n W_{ni}^2(x)}} \quad \text{and} \quad T_{n,x}^{\text{het}} = \frac{\hat{m}_n(x) - \mathbb{E}\hat{m}_n(x)}{\sqrt{\sum_{i=1}^n W_{ni}^2(x) \hat{\epsilon}_i^2}}$$

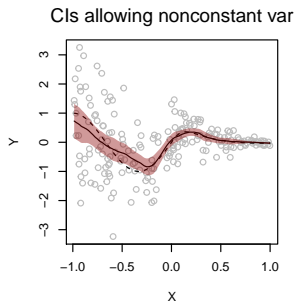
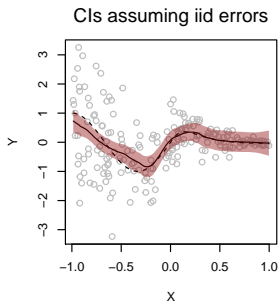
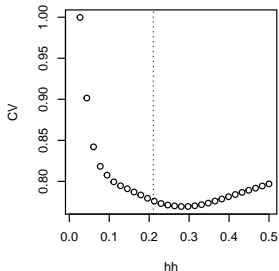
```

m <- function(x){sin(3* pi * x / 2)/(1 + 18 * x^2*(sign(x) + 1))}
n <- 200
X <- runif(n,-1,1)
sigma <- (1.5 - X)^2/4
Y <- m(X) + rnorm(n,0,sigma)

```

The asymptotic Normality of the pivots suggests the pointwise CIs

$$\left[\hat{m}_n(x) \pm z_{\alpha/2} \hat{\sigma}_n \sqrt{\sum_{i=1}^n W_{ni}^2(x)} \right] \quad \text{and} \quad \left[\hat{m}_n(x) \pm z_{\alpha/2} \sqrt{\sum_{i=1}^n W_{ni}^2(x) \hat{\varepsilon}_i^2} \right].$$



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Consider the constant variances case $\sigma_1^2 = \dots = \sigma_n^2 = \sigma^2 \in (0, \infty)$.

Assume \hat{m}_n is undersmoothed and pretend $\mathbb{E}\hat{m}_n(x)$ equals $m(x)$ (i.e. ignore bias).

A pivot for building confidence bands

Define

$$T_n = \sup_{x \in [0,1]} \left| \frac{\hat{m}_n(x) - \mathbb{E}\hat{m}_n(x)}{\hat{\sigma}_n \sqrt{\sum_{i=1}^n W_{ni}^2(x)}} \right|$$

and denote by G_n the distribution of T_n .

Exercise: Propose a confidence band for $m(x)$, $x \in [0, 1]$ of the form

$$\mathcal{B}_{n,\alpha} = \{(x, y) : l_{n,\alpha}(x) \leq y \leq u_{n,\alpha}(x), x \in [0, 1]\}$$

assuming the distribution G_n is known.

Asymptotic result of Sun and Loader (1994) [3], [4].

For $Z_1, \dots, Z_n \stackrel{\text{ind}}{\sim} \text{Normal}(0, 1)$, we have

$$P\left(\sup_{x \in [0,1]} \left| \sum_{i=1}^n M_{ni}(x) Z_i \right| > c\right) \approx 2(1 - \Phi(c)) + \frac{\kappa_0}{\pi} e^{-c^2/2}$$

for large c and large enough n , where $\kappa_0 = \int_0^1 \sqrt{\sum_{i=1}^n \left[\frac{\partial}{\partial x} M_{ni}(x)\right]^2} dx$.

The above result is sometimes called a *tube formula*.

Exercise:

- Find $M_{ni}(x)$, $i = 1, \dots, n$, such that we may write

$$T_n = \sup_{x \in [0,1]} \left| \sum_{i=1}^n M_{ni}(x) (\varepsilon_i / \hat{\sigma}_n) \right|$$

- Propose a conf. band for $m(x)$, $x \in [0, 1]$ based on the above result.

Approximation to κ_0

To compute κ_0 in practice is hard. We can get an approximation to it as

$$\kappa_0 \approx \sum_{j=1}^{N-1} \sqrt{\sum_{i=1}^n [M_{ni}(x_{j+1}) - M_{ni}(x_j)]^2}$$

given a grid of values $x_1, \dots, x_N \in [0, 1]$, for some large N .

Exercise:

- 1 Justify the above approximation to κ_0 .
- 2 Construct the asymptotic $(1 - \alpha)100\%$ confidence band

$$\left\{ (x, y) : y \in \left[\hat{m}_n(x) \pm c_\alpha \hat{\sigma}_n \sqrt{\sum_{i=1}^n W_{ni}^2(x)} \right], x \in [0, 1] \right\}$$

on a simulated data set, where c_α is from the Sun and Loader method.

Consider the case in which $\sigma_1^2, \dots, \sigma_n^2 \in (0, \infty)$ are heteroscedastic.

Assume \hat{m}_n is undersmoothed and pretend $\mathbb{E}\hat{m}_n(x)$ equals $m(x)$ (i.e. ignore bias).

A pivot for building confidence bands under heteroscedasticity

Define

$$T_n^{\text{het}} = \sup_{x \in [0,1]} \left| \frac{\hat{m}_n(x) - \mathbb{E}\hat{m}_n(x)}{\sqrt{\sum_{i=1}^n W_{ni}^2(x) \hat{\epsilon}_i^2}} \right|$$

and denote by G_n^{het} the distribution of T_n^{het} .

Exercise: Propose a confidence band for $m(x)$, $x \in [0, 1]$ of the form

$$\mathcal{B}_{n,\alpha}^{\text{het}} = \{(x, y) : l_{n,\alpha}^{\text{het}}(x) \leq y \leq u_{n,\alpha}^{\text{het}}(x), x \in [0, 1]\}$$

assuming the distribution G_n^{het} is known.

Exercise:

- 1 To build a conf. band à la Sun and Loader, we need $\bar{M}_{ni}(x)$ such that

$$T_n^{\text{het}} = \sup_{x \in [0,1]} \left| \sum_{i=1}^n \bar{M}_{ni}(x) Z_i \right|$$

for some rvs $Z_1, \dots, Z_n \stackrel{\text{ind}}{\sim} \text{Normal}(0, 1)$. Find $\bar{M}_{ni}(x)$, $i = 1, \dots, n$.

- 2 Construct the asymptotic $(1 - \alpha)100\%$ confidence band

$$\left\{ (x, y) : y \in \left[\hat{m}_n(x) \pm c_\alpha \sqrt{\sum_{i=1}^n W_{ni}^2(x) \hat{\varepsilon}_i^2} \right], x \in [0, 1] \right\}$$

on a simulated data set, where c_α is from the S&L method with κ_0 based on

$$\hat{M}_{ni}(x) = \frac{W_{ni}(x) |\hat{\varepsilon}_i|}{\sqrt{\sum_{i=1}^n W_{ni}^2(x) \hat{\varepsilon}_i^2}}.$$

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Consider the constant variances case $\sigma_1^2 = \dots = \sigma_n^2 = \sigma^2 \in (0, \infty)$.

Residual bootstrap for nonparametric regression

- 1 Draw $\varepsilon_1^*, \dots, \varepsilon_n^*$ w/repl from $\hat{\varepsilon}_i = Y_i - \hat{m}_n(X_i)$, $i = 1, \dots, n$.
- 2 Set $Y_i^* = \hat{m}_n(X_i) + \varepsilon_i^*$, $i = 1, \dots, n$.
- 3 Compute $\hat{m}_n^*(x) = \sum_{i=1}^n W_{ni}(x) Y_i^*$ and $\hat{\sigma}_n^*$ based on $(X_1, Y_1^*), \dots, (X_n, Y_n^*)$.
- 4 Compute bootstrap version of $T_{n,x}$ given by

$$T_{n,x}^* = \frac{\hat{m}_n^*(x) - \mathbb{E}_* \hat{m}_n^*(x)}{\hat{\sigma}_n^* \sqrt{\sum_{i=1}^n W_{ni}^2(x)}}$$

Exercise: Show that $\hat{m}_n^*(x) - \mathbb{E}_* \hat{m}_n^*(x) = \sum_{i=1}^n W_{ni}(x) \varepsilon_i^*$, provided $\sum_{i=1}^n \hat{\varepsilon}_i = 0$.

Consider the case in which $\sigma_1^2, \dots, \sigma_n^2 \in (0, \infty)$ are heteroscedastic.

Wild bootstrap for nonparametric regression

- 1 Generate indep. bootstrap residuals $\varepsilon_1^{*W}, \dots, \varepsilon_n^{*W}$ satisfying $\mathbb{E}_*[\varepsilon_i^{*W}] = 0$, $\mathbb{E}_*[(\varepsilon_i^{*W})^2] = \hat{\varepsilon}_i^2$, and $\mathbb{E}_*[(\varepsilon_i^{*W})^3] = \hat{\varepsilon}_i^3$, where $\hat{\varepsilon}_i = Y_i - \hat{m}_n(X_i)$, $i = 1, \dots, n$.
- 2 Set $Y_i^{*W} = \hat{m}_n(X_i) + \varepsilon_i^{*W}$, $i = 1, \dots, n$.
- 3 Compute $\hat{m}_n^{*W}(x) = \sum_{i=1}^n W_{ni}(x) Y_i^{*W}$.
- 4 Compute bootstrap version of $T_{n,x}^{\text{het}}$ given by

$$T_{n,x}^{\text{het}*} = \frac{\hat{m}_n^{*W}(x) - \mathbb{E}_* \hat{m}_n^{*W}(x)}{\sqrt{\sum_{i=1}^n W_{ni}^2(x) (\hat{\varepsilon}_i^{*W})^2}}$$

Consider constructing bootstrap versions of the pivots T_n and T_n^{het} .

Residual/wild bootstrap versions of T_n and T_n^{het}

Given a grid x_1, \dots, x_N of values in $[0, 1]$, define

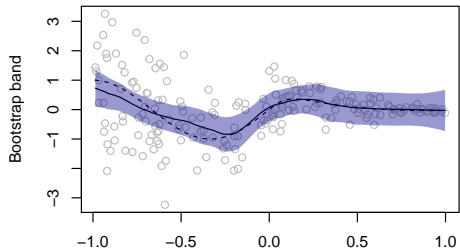
$$T_n^* = \max_{1 \leq j \leq N} \left| \frac{\sum_{i=1}^n W_{ni}(x_j) \varepsilon_i^*}{\hat{\sigma}_n^* \sqrt{\sum_{i=1}^n W_{ni}^2(x_j)}} \right| \quad \text{and} \quad T_n^{\text{het}*} = \max_{1 \leq j \leq N} \left| \frac{\sum_{i=1}^n W_{ni}(x_j) \varepsilon_i^{*W}}{\sqrt{\sum_{i=1}^n W_{ni}^2(x_j) (\hat{\varepsilon}_i^{*W})^2}} \right|$$

and denote by \hat{G}_n and \hat{G}_n^{het} the dists of T_n^* and $T_n^{\text{het}*}$ conditional on the data.

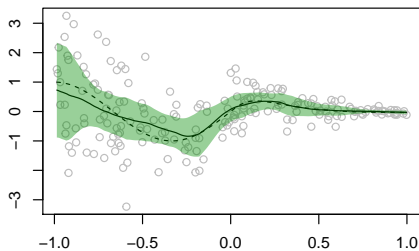
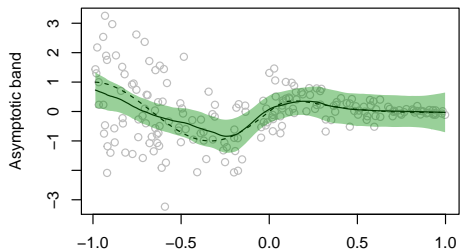
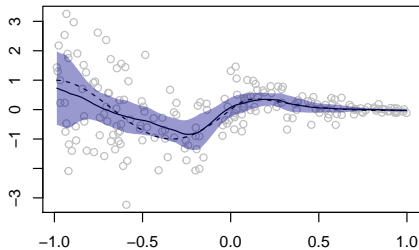
Exercise:


- 1 Give the form of $(1 - \alpha)100\%$ bootstrap confidence bands under constant variances and heteroscedasticity.
- 2 Demonstrate on simulated data.

assuming iid errors





allowing heteroscedastic errors




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