STAT 515 - Chapter 7 Supplement

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S7 - The Math Behind Confidence Intervals

Chapter 7 discusses several confidence intervals, including the interval for one population mean and for one population variance. All of those follow directly from the sampling distributions discussed in the Supplement to Section 6.3.

S 7.3 The Confidence Interval for μ

Sections 7.2 and 7.3 in the text discuss the confidence interval for a mean. Since we'll never know the population mean in practice, the tool for making one of these intervals is the t-relationship from S6.3.3. If we have a simple random sample from a large enough normally distributed population (a.k.a. an independent and identically distributed normal random sample), then we know that:

$$t_{df=n-1} = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

Say that we want to make a $1 - \alpha$ confidence interval. We picture this by saying we want to capture the middle $1 - \alpha$ (say 95%), so that there is $\alpha/2$ (say 2.5%) in each end. We denote the t-values that do this by $\pm t_{\alpha/2}$. For example, in Example 7.4 on page 329 the values are -2.571 and +2.571 to go with 95% for 5 degrees of freedom. We could write this mathematically as:

$$P[-t_{\alpha/2} \;\leq\; \frac{\bar{x} - \mu}{s/\sqrt{n}} \;\leq\; t_{\alpha/2}\,] = 1 - \alpha \qquad (S1)$$

Just as we can use the t-distribution to form a confidence interval for the mean of a population, we can use the χ^2 and F distributions to make confidence intervals for the variance of a population, or the ratio of two variances. The logic is the same in both cases: use the sampling distribution to form a probability statement containing just the one unknown parameter, and then solve for that parameter.

The Confidence Interval for σ or σ^2

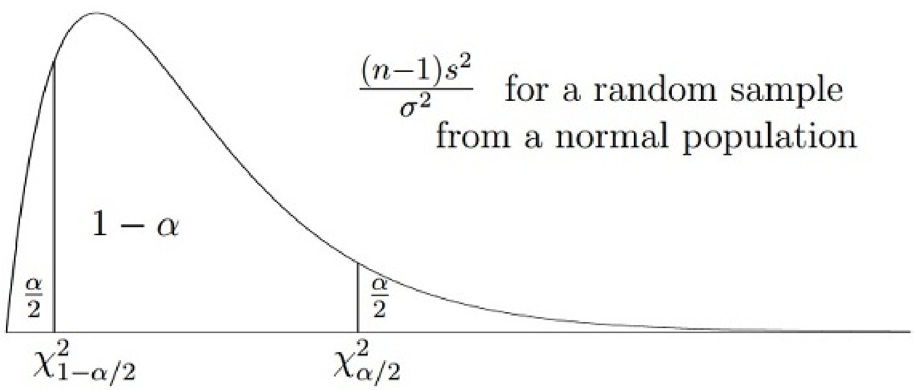
Section 7.6 in the text discusses the confidence interval for one population variance. We can see how that works by referring back to section S6.4.2. There we saw that if the random sample was drawn from a population that was normally distributed, then

$$\chi^2_{df=n-1} = \frac{(n-1)s^2}{\sigma^2}$$

If we choose $\chi^2_{\alpha/2}$ to be the value such that $P(\chi^2_{df=n-1} \geq \chi^2_{\alpha/2}) = \frac{\alpha}{2}$ and $\chi^2_{1-\alpha/2}$ to be the value such that $P(\chi^2_{df=n-1} \leq \chi^2_{1-\alpha/2}) = \frac{\alpha}{2}$ we get the following:

$$P[\ \chi^2_{1-\alpha/2}\ \leq\ \frac{(n-1)s^2}{\sigma^2}\ \leq\ \chi^2_{\alpha/2}\ ] = 1-\alpha \qquad (S1)$$

This is illustrated in the figure below.



We can now solve the inequality in S1 for σ^2 to get the confidence interval.

$$P\left[\ \chi^2_{1-\alpha/2}\ \leq\ \frac{(n-1)s^2}{\sigma^2}\ \leq\ \chi^2_{\alpha/2}\ \right]$$

$$= P\left[\ \frac{1}{\chi^2_{1-\alpha/2}}\ \geq\ \frac{\sigma^2}{(n-1)s^2}\ \geq\ \frac{1}{\chi^2_{\alpha/2}}\ \right]$$

$$= P\left[\ \frac{1}{\chi^2_{\alpha/2}}\ \leq\ \frac{\sigma^2}{(n-1)s^2}\ \leq\ \frac{1}{\chi^2_{1-\alpha/2}}\ \right]$$

$$= P\left[\ \frac{(n-1)s^2}{\chi^2_{\alpha/2}}\ \leq\ \sigma^2\ \leq\ \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}\ \right] = 1-\alpha$$

The $(1-\alpha)100\%$ confidence interval for the population variance is thus:

$$\left(\ \frac{(n-1)s^2}{\chi^2_{\alpha/2}}\ ,\ \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}\ \right) \qquad (S2)$$

This can be changed to a confidence interval for the population standard deviation simply by taking the square root of both sides.

$$\left(\sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2}}}\ ,\ \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}} \right) \qquad (S3)$$

Say we are interested in the standard deviation of a certain population. A sample of size $n = 12$ ($df = 11$) is gathered, s^2 is found to be 20.2, the q-q plot for the data looks fairly normal, and it is desired to construct a 95% confidence interval. Using S3, we see that we need to determine $\chi^2_{\alpha/2}$ and $\chi^2_{1-\alpha/2}$, where $\alpha/2 = \frac{1-95\%}{2} = \frac{.05}{2} = 0.025$ and $1-\alpha/2 = 1-0.025 = 0.975$. From Table VII we see that the values are $\chi^2_{0.025} = 21.9200$ and $\chi^2_{0.975} = 3.81575$. Plugging these values into S3 gives: $(\sqrt{10.137}, \sqrt{58.232}) = (3.18, 7.63)$.

The Confidence Interval for $\frac{\sigma_x^2}{\sigma_y^2}$

A confidence interval for the ratio of two variances could be constructed in the same way as one for a single variance. From S6.4.4 we see that if two samples $x_1, x_2, \ldots x_{n_x}$ and $y_1, y_2, \ldots y_{n_y}$ are drawn randomly from two populations that seem normal, then:

$$F_{df_x=n_x-1, df_y=n_y-1} = \frac{\frac{s_x^2}{s_y^2}}{\frac{\sigma_x^2}{\sigma_y^2}}$$

If we choose $F_{\alpha/2}$ to be the value such that $P(F_{df_x=n_x-1,df_y=n_y-1} \geq F_{\alpha/2}) = \frac{\alpha}{2}$ and $F_{1-\alpha/2}$ to be the value such that $P(F_{df_x=n_x-1,df_y=n_y-1} \leq F_{1-\alpha/2}) = \frac{\alpha}{2}$ we get the following:

$$P\left[F_{1-\alpha/2} \ \leq\ \frac{\frac{s_x^2}{s_y^2}}{\frac{\sigma_x^2}{\sigma_y^2}} \ \leq\ F_{\alpha/2} \right] = 1-\alpha \qquad (S4)$$

Solving for $\frac{\sigma_x^2}{\sigma_y^2}$ then gives us the $(1-\alpha)100\%$ confidence interval:

$$\left(\frac{\frac{s_x^2}{s_y^2}}{F_{\alpha/2}}\ ,\ \frac{\frac{s_x^2}{s_y^2}}{F_{1-\alpha/2}} \right) \qquad (S5)$$

It might be good practice to see if you can work out the steps between S4 and S5.

Note that the text only gives the table for finding the $F_{\alpha/2}$ values, and not the $F_{1-\alpha/2}$ ones. One way of getting around this would be to use SAS or R; just remember that SAS and R give the area in the lower end of the table, while the text (and the formulas above) use the upper end. Another way is to use the following relationship:

$$F_{1-\alpha/2,df_x=n_x-1,df_y=n_y-1} = \frac{1}{F_{\alpha/2,df_x=n_y-1,df_y=n_x-1}}$$

That is, to find the $F_{0.975}$ you flip the degrees of freedom and take $1/F_{0.025}$. So, if you wanted a 95% confidence interval when $n_x = 6$ and $n_y = 5$ the two values you would use are 9.36 for $F_{0.025}$ (straight from the table for df of 5 and 4) and $1/7.39 = 0.135$ for $F_{0.975}$ (one over the value from the table for the reversed df of 4 and 5).

Robustness... What if the Data Isn't Normal?

A statistical procedure is called robust if it performs well even when its assumptions aren't met. In the case of using the t-distribution to make inferences about the mean, the χ^2-distribution for the variance, and the F-distribution for two variances, we need to assume that the initial populations were normally distributed. The procedures that use the t-distribution are fairly robust however. That is, the procedures involving the use of the t-distribution to make inferences about μ work fairly well even when the data isn't normally distributed. In general it will work well for small sample sizes ($n \leq 30$) even if their are some doubts about the q-q plot. For large sample sizes it will work well for all but the worst q-q plots. The procedures discussed in S6.1 and S6.2 for making inferences about population variances are not robust at all. If there are any questions about the q-q plots, they should not be used. It is important to note that in Chapters 10, 11, and 13 we will see other uses of the χ^2 and F distributions that are robust. A particular distribution is never robust or non-robust, robustness is a property of the entire procedure that you are attempting.

In this course we are only covering some of the most common and basic methods for making inferences about a population. A variety of other methods are discussed in STAT 518 - Nonparametric Statistical Methods. A method is called nonparametric if it does not depend on the assumptions that the data follows a particular distribution (like the normal distribution). There are two reasons for not simply always using nonparametric methods. One is that they are somewhat more complicated to explain (see Section 14.2 on the CD with the text for example). The second reason is that while the nonparametric tests are generally better when the assumptions are badly violated, the standard methods we are learning here are better when the assumptions are met.