

STAT 515 - Section 6.3 Supplement

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S6.3 - More on Sampling Distributions: The t , χ^2 , and F Distributions

As we saw in Section 6.3, the normal distribution plays a pivotal roll in describing how the sample mean $\bar{x}$ will behave when you have a random sample $x_1, x_2, \dots, x_n$. Unfortunately the central limit theorem only applies when the sample size is large. Additionally, it only tells us about the sampling distribution of the sample mean, and not about the sampling distribution of the sample variance s^2 . These limitations can be overcome if we can believe that the sample was taken from a population that was normal to begin with. That is, if we apply the methods in Section 5.4 and verify the data is normal, we can get the sampling distribution for $\bar{x}$ when n is small, and can also get the sampling distribution for s^2 .

S6.3.1 - $\bar{x}$ and the Normal Distribution

A fact that is proved in STAT 512 is: if the random sample is drawn from a population that follows a normal distribution, then $Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ is exactly standard normal. In other words, if the base population is already normal, the central limit theorem result applies even when $n = 1$! The only difficulty in this is that we rarely, if ever, know the value of the parameter σ . Because of this we can't use this fact directly.

S6.3.2 - s^2 and the χ^2 (chi-squared) Distribution

The χ^2 distribution can be defined as follows. If $z_1, z_2, \dots, z_n$ are independent and each follows the standard normal distribution, then

$$X^2 = z_1^2 + z_2^2 + \dots + z_n^2$$

follows the χ^2 distribution with n degrees of freedom (df). The table for this distribution (and a typical picture of it) can be found in TABLE IV on pages 818. This distribution is skewed to the right, has mean = df, variance = 2df, and takes all values 0 and higher. (The normal on the other hand takes all positive and negative values.)

The usefulness of this distribution becomes a bit clearer if we again consider the random sample $x_1, x_2, \dots, x_n$ from a normal distribution. Looking at the formula for s^2 :

$$s^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1}$$

we can see that we are squaring a bunch of independent normal random variables (the x_i) and summing them up. The only reason that this isn't a χ^2 random variable is that they aren't standard normal, and we are dividing by $n-1$.

By multiplying both sides of the above equation by $n - 1$ and dividing by σ^2 we get the following:

$$\frac{(n - 1)s^2}{\sigma^2} = \sum_{i=1}^n \left(\frac{x_i - \bar{x}}{\sigma} \right)^2$$

If the $\bar{x}$ on the right side of the equation were replaced by μ , then we would be summing up a bunch of $z = \frac{x_i - \mu}{\sigma}$ and it would be the sum of n standard normals, making a χ^2 random variable with n degrees of freedom. Because we are using $\bar{x}$ instead of μ we lose one degree of freedom, and so:

$$\chi_{df=n-1}^2 = \frac{(n - 1)s^2}{\sigma^2} \tag{S1}$$

where the df in the subscript is the number of degrees of freedom.

If the data come from a sample that is normal, we know that the left hand side of equation S1 behaves as a χ^2 random variable, we know the value $n - 1$ because it is based on the sample size, and we know the value of s^2 because we can calculate it from the data. If we solve this equation for σ^2 we could then get information about this unknown population parameter. We will discuss this more in Chapters 7 and 8.

Table IV on pages 818-819 of the text gives several of the values of the χ^2 random variable for a variety of degrees of freedom. Each row of this table corresponds to a different number of degrees of freedom (df). Remember that you need to look at $df = n - 1$ if you are using the sample variance to investigate the population variance. The rest of the table is set up the opposite of the normal table. The values along the top are the probabilities (the areas that are shaded in on the figure) and the body of the table contains the χ^2 values that go with those probabilities. Because each df needs its own row, no table can possibly contain all of the possible values. (A normal table can because we can change any normal to a standard normal, there is no such simplification for the χ^2 .)

Say we wanted to know $P(\chi_{df=8}^2 \leq 1.344)$. Looking at the $df = 8$ row of Table IV, we see that this corresponds to a probability of 0.995. The probability is for greater than 1.344 though, so we have to take $1 - 0.995$ and get a value of 0.005. Many of the values are not in the table however. If we wanted to know $P(\chi_{df=8}^2 \geq 2.5)$ the closest values in the table are 2.179 and 2.733. Because of this, all we can say about the probability is that it is between 0.950 and 0.975. If an exact value is needed then a computer package such as SAS or R would be used instead. We can also use the table in the reverse direction. The c_0 such that $P(\chi_{df=8}^2 \geq c_0) = 0.010$ is 20.0902. The c_0 such that $P(\chi_{df=8}^2 \leq c_0) = 0.010$ is 1.646482.

S6.3.3 - $\bar{x}$, s , and Student's t-Distribution

As noted in S6.3.1, the difficulty with the central limit theorem is that it requires us to know σ . The tool we use to deal with this is the t distribution that was

discovered in 1908 by chemist William Gosset at the Guinness brewery in Ireland. Because he didn't want employees at other breweries to know that he found statistics useful, he published his results under the pseudonym *Student*. Hence, the distribution is often known as Student's t distribution. A t-distribution is formed by dividing a standard normal by a χ^2 over its degrees of freedom, where the normal and the χ^2 are independent.

$$t_{df=n} = \frac{Z}{\sqrt{\frac{\chi_{df=n}^2}{n}}} \quad (S2)$$

At first this seems to be more than a little bit out of nowhere. A fact proved in STAT 512 sheds some light on why it is useful however. If the sample $x_1, x_2, \dots, x_n$ is independent, then its sample mean $\bar{x}$ and sample variance s^2 are independent! If the sample comes from a normal distribution, S6.3.1 showed that $\bar{x}$ is related to a standard normal distribution, and S6.3.2 showed that s^2 is related to a χ^2 distribution. Combining these previous results gives:

$$t_{df=n-1} = \frac{Z}{\sqrt{\frac{\chi_{df=n-1}^2}{n-1}}} = \frac{\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)s^2}{\sigma^2}}}$$

By cancelling the $n-1$ terms in the denominator, applying the square root, multiplying both the numerator and denominator by one over the denominator, and cancelling, we get

$$t_{df=n-1} = \frac{\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{s^2}{\sigma^2}}} = \frac{\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}}{\frac{s}{\sigma}} = \frac{\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \frac{\sigma}{s}}{\frac{s}{\sigma}} = \frac{\bar{x}-\mu}{s/\sqrt{n}} \quad (S3)$$

Just as we could solve equation S1 to find out information about the population variance, we can solve equation S3 to find out information about the population mean, if the sample comes from a population that follows the normal distribution.

This usage is discussed more in Section 7.3. One useful fact about the t distribution is that it becomes very similar to the standard normal distribution as the sample size n increases.

Many tables for the t distribution stop at 30 degrees of freedom and simply refer the user to a standard normal table. Our table III on page 817 continues on past 30, but does not give all the values. Notice that the values change very little from one row to the next after about row eighteen. If you go to the bottom row, all of the values should be recognizable from the normal table.

S6.3.4 - Two variances and the F-Distribution

The final sampling distribution we will be concerned with is the F-distribution. The F-distribution is defined by:

$$F_{df_x=n_x, df_y=n_y} = \frac{\frac{X_x^2}{n_x}}{\frac{X_y^2}{n_y}} \quad (S4)$$

where X_x^2 and X_y^2 are independent χ^2 random variables with n_x and n_y degrees of freedom respectively. Because it is formed by using two χ^2 random variables, the F-distribution has two separate degrees of freedom, one for the numerator and one for the denominator. This makes the F tables even more complicated than the χ^2 or t tables.

The formula for the F-distribution again looks out of nowhere, until you recognize that we could get this formula by comparing two variances. Say we have independent random samples from two populations, call them $x_1, x_2, \dots, x_{n_x}$ and $y_1, y_2, \dots, y_{n_y}$. We could then write:

$$F_{df_x=n_x-1, df_y=n_y-1} = \frac{\frac{(n_x-1)s_x^2}{\sigma_x^2}}{\frac{(n_y-1)s_y^2}{\sigma_y^2}}$$

Canceling and then inverting the fractions, we get:

$$F_{df_x=n_x-1, df_y=n_y-1} = \frac{\frac{s_x^2}{\sigma_x^2}}{\frac{s_y^2}{\sigma_y^2}} = \frac{\frac{s_x^2}{s_y^2}}{\frac{\sigma_x^2}{\sigma_y^2}} \quad (S5)$$

Equation number S5 thus allows us to compare the variances of two different populations, if we can assume both populations are normally distributed. Tables V, VI, VI, and VIII (pages 820-827) give the values of the F-distribution for various combinations of degrees of freedoms and areas under the curve. The F-distribution occurs not only for comparing two variances (more in Section 9.6), but in Section 10.2 we will see that it is useful for comparing more than two means, and in Chapter 11 that it is useful for predicting one variable from another.

One final fact that we will encounter later concerns the relationship between the t-distribution and the F-distribution. Turn back to formula S2, and square the numerator and the denominator. The denominator becomes the same as the denominator in S5. The numerator becomes a Z^2 , which is just a χ^2 with one degree of freedom. We thus get the following result.

$$(t_{df=n-1})^2 = F_{df_x=1, df_y=n-1}$$

This result means that in some cases in Chapter 11 we will be able to work with either a t-distribution or an F-distribution and get the same result.