Chapter 1: #56 (define the sample space and relevant events, and give the probabilities of the sample points)
For the sample space use the notation (oldest youngest) with G=girl and B=boy.
{(BB),(GG),(GB),(BG)} with each having probability ¼.

\[
P(\text{Both Girls }| \text{ Oldest is a Girl}) = \frac{P(\text{Both Girls } \cap \text{ Oldest is a Girl})}{P(\text{Oldest is a Girl})} = \frac{P(\text{GG})}{P(\text{GG} \cup \text{GB})} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4}} = \frac{1}{2}
\]

\[
P(\text{Both Girls }| \text{ At least one G}) = \frac{P(\text{Both Girls } \cap \text{ At least one G})}{P(\text{At least one G})} = \frac{P(\text{GG})}{P(\text{GG} \cup \text{GB} \cup \text{BG})} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} = \frac{1}{3}
\]

#65: By the addition rule we have \(P((A \cap B) \cup (A \cap B^C)) = P(A) + P(B) - P(A \cap B).
As (A \cap B) and (A \cap B^C) are disjoint, this becomes: \(P(A) = P(A \cap B) - P(A)P(B) = P(A) - P(A)P(B)\)
And \(P(A \cap B^C) = P(A)P(B^C)\) is the definition of A and B^C being independent.

By the addition rule we have \(P((A \cap B^C) \cup (A^C \cap B)) = P(A \cap B^C) + P(A^C \cap B) - 0\)
As (A \cap B^C) and (A^C \cap B) are disjoint, this becomes: \(P(B^C) = P(A \cap B^C) + P(A^C \cap B)\)
As A and B^C are independent this becomes: \(P(B^C) = P(A) - P(A)P(B) = P(A) - P(A)P(B)\)
And \(P(A^C \cap B) = P(A)P(B)\) is the definition of A^C and B being independent.

#66: \(P(\emptyset \cap A) = P(\emptyset) = 0 = P(A) = P(\emptyset) P(A)\)

Chapter 1: #38: There are 6! ways of arranging six blocks, but we don’t care about the order of the three reds or three greens, so \(\binom{6}{3} = \frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3! \cdot 3 \cdot 2 \cdot 1} = \frac{120}{6} = 20\)

Similarly: \(\binom{6}{3,3,3} = \frac{9!}{3!3!3!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!}{3! \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = \frac{60480}{36} = 1680\)

#42: \(\binom{11}{4,3,3,1} = \frac{11!}{4!3!3!1!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 3! \cdot 3! \cdot 1 \cdot 1} = \frac{1663200}{36} = 46200\)

#57: Let A, B, and C be the three respective cabinets. Let S be that a silver was found in the first drawer. Finally, notice the only way a silver can be in both is to get drawer B, so what we want is \(P(B|S)\). By Bayes’ rule we get: \(P(B|S) = P(S|B) / \left[ P(S|A) + P(S|B) + P(S|C) \right] = 1 / (0 + 1 + \frac{1}{2}) = 2/3\)
Chapter 1: #17 (you may treat it as a binomial or hypergeometric; give the formula for \( p=0.2 \); use R to make the graph, plotting the values \( p=0,0.05,0.1,0.15,0.2, \) and 0.25)

The population size is 100, the number of defectives is \( m \), the number sampled is 4, and the number of defective founds is \( x \). Note in this case that \( p=m/100 \) and so \( m=100p \).

So, \( P[\text{accept lot}]=P[0 \text{ defective}] = \begin{pmatrix} 100p \\ 0 \end{pmatrix} \begin{pmatrix} 100-100p \\ 4 \end{pmatrix} \begin{pmatrix} 100 \\ 4 \end{pmatrix} \)

For \( p=0.2 \) this is \( \begin{pmatrix} 20 \\ 0 \end{pmatrix} \begin{pmatrix} 80 \\ 4 \end{pmatrix} = \frac{80!}{76!4!} = \frac{80 \cdot 79 \cdot 78 \cdot 77}{96!4!} \cdot \frac{100!}{100!} \cdot \frac{100\cdot 99 \cdot 98 \cdot 97}{96!4!} = 0.4033382 \)

\[
\text{percdef<-c(0,0.05,0.10,0.15,0.2,0.25)}
\]
\[
\text{numbad<-100*percdef}
\]
\[
\text{numgood<-100-numbad}
\]
\[
\text{probaccept<-dhyper(0,numbad,numgood,4)}
\]
\[
\text{plot(percdef,probaccept,type="l",xlim=c(0,.25),ylim=c(0,1))}
\]
Chapter 1: #18a  So $P[\text{at least 1 defective}]=0.9$ is the same as $P[0 \text{ defective}]=0.1$. Using the hypergeometric we get:

$$
\binom{10}{0}\left(\frac{990}{m}\right) = \frac{990!}{m!(990-m)!} \cdot \frac{1000!}{1000!} = \frac{990!}{m!(1000-m)!} = \frac{1000-m}{1000!} \cdots (991-m)
$$

There is no nice way to solve this directly.

Using brute force in R:

```r
f<-function(m,n=1000,k=10){
  prod((n-m):(n-m-k+1))/prod(n:(n-k+1))
}
values<-cbind( m=1:1000 , prob=sapply(1:1000,f) )
round(values,4)

[204,]  204 0.1009
[205,]  205 0.0997
```

So we get 205.

Or, if you say that 1000 is big relative to the $m$ then we have approximately

$$
0.1 = \frac{(1000-m)\cdots(991-m)}{1000\cdots991} \approx \frac{(1000-m)^{10}}{1000^{10}}
$$

$$
\Rightarrow 0.1(1000)^{10} = (1000-m)^{10}
$$

$$
\Rightarrow 0.1(1000)^{10} = 1000 - m
$$

$$
\Rightarrow m = 1000 - [0.1(1000)^{10}]^{0.1} = 205.6718
$$

So approximately 206 (which is pretty close to the actual value of 205).

#35a: \[ \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(n-r)!} = \binom{n}{n-r} \]

When you select $r$ out of $n$ without replacement the remainder is a sample of $n-r$.

#36: Using proposition B on page 12 it is just

$$
\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = \frac{7\cdot6\cdot5\cdot4!}{3\cdot2\cdot1\cdot4!} = \frac{210}{6} = 35
$$
Chapter 2: #1 (also calculate the mean and variance)

\[ \mu = E(X) = \sum x p(x) = 0(0.25) + 1(0.125) + 2(0.125) + 3(0.5) = 0.125 + 0.25 + 1.5 = 1.875 \]
\[ \sigma^2 = Var(X) = E[(X - \mu)^2] = \sum (x - \mu)^2 p(x) \]
\[ = (0 - 1.875)^2(0.25) + (1 - 1.875)^2(0.125) + (2 - 1.875)^2(0.125) + (3 - 1.875)^2(0.5) \approx 1.609 \]

#11: Want the largest \( k \) for which the ratio is increasing.

\[
\text{ratio} = \frac{P(X = k)}{P(X = k - 1)} = \frac{\binom{n}{k} p^k (1-p)^{n-k}}{\binom{n}{k-1} p^{k-1} (1-p)^{n-(k-1)}} = \frac{n!}{k!(n-k)!} \frac{p^k (1-p)^{n-k}}{p^{k-1} (1-p)^{n-(k-1)}} = \frac{n!}{(k-1)!(n-k+1)!} \frac{p^{k-1} (1-p)^{n-(k-1)}}{p^{k-1} (1-p)^{n-(k-1)}} = \frac{1}{k} \frac{p}{(1-p)} = \frac{(n-k+1)p}{k(1-p)} > 1
\]

\[
\Rightarrow (n-k+1)p > k(1-p) \quad \Rightarrow \quad np - kp + p > k - kp \quad \Rightarrow \quad np + p > k
\]

So, the largest integer less than \( np + p \)
Chapter 4: #45a  
E(Z)=E(\alpha X+(1-\alpha)Y)=\alpha E(X)+(1-\alpha)E(Y)=\alpha \mu+(1-\alpha)\mu=\mu

b: Var(Z)=Var(\alpha X+(1-\alpha)Y)=\alpha^2 Var(X)+(1-\alpha)^2 Var(Y)=\alpha^2 \sigma_X^2+(1-\alpha)^2 \sigma_Y^2

Now we need to take the derivative and set it equal to zero to find the extrema:

$$\frac{\partial \text{Var}(Z)}{\partial \alpha} = 2\alpha \sigma_X^2 + 2(1-\alpha)(-1)\sigma_Y^2 = 2\alpha \sigma_X^2 + 2\alpha \sigma_Y^2 - 2\sigma_Y^2 = 0 \quad \Rightarrow \quad \alpha = \frac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}$$

To verify it’s a minima we need the second derivative to be positive:

$$\frac{\partial^2 \text{Var}(Z)}{\partial \alpha^2} = 2\sigma_X^2 + 2\sigma_Y^2 > 0$$

We should also check the endpoint values of 0 and 1. Var(Z) at \alpha=0 is \sigma_Y^2, Var(Z) at \alpha=1 is \sigma_X^2. The Var(Z) at the value we found above is \(\frac{\sigma_Y^2 \sigma_X^2}{\sigma_X^2 + \sigma_Y^2}\) which is less than either \sigma_X^2 or \sigma_Y^2.

c: Var( (X+Y)/2 ) = 0.25 \sigma_X^2 + 0.25 \sigma_Y^2 

\((1/4)\sigma_X^2 + (1/4)\sigma_Y^2 < \sigma_Y^2 \rightarrow (1/4)\sigma_X^2 < (3/4)\sigma_Y^2 \rightarrow \sigma_X^2 < 3\sigma_Y^2\)

Similarly \ \sigma_Y^2 < 3\sigma_X^2.

So it is better to use the average when \ \sigma_X^2 < 3\sigma_Y^2 \ and \ \sigma_Y^2 < 3\sigma_X^2.

Homework 7 – Due 9/21

Chapter 2: #31a
First we need to convert to the appropriate amount of time \(\lambda = 2/\text{hour} = 1/3 / 10\text{ minutes}.
Secondly notice that P[rings]=P[at least one occurrence]=1-P[no occurrences]=1-(1/3)^0 e^{-1/3/0!} \approx 1-.7165=0.2835

Also, a: The population size 4,000,000 is very large, and the sample size is very small relative to it, so that the binomial and hypergeometric distributions will be very similar.

b: \(P[X = 0] = \binom{100}{0} (0.04)^0 (1 - 0.04)^{100-0} = 0.96^{100} \approx 0.0169 = 1.69\% \)

c: This would be the expected value of a geometric distribution = 1/p = 1/0.04 = 25.

d: This is a negative binomial with r=2, so

\(P[X = 10] = \binom{10-1}{2-1} (0.04)^2 (1 - 0.04)^{10-2} = 9 \cdot 0.04^2 \cdot 0.96^8 \approx 0.0104 = 1.04\% \)