Today

• Homework
• Order Statistics (cont.)
• More on Expected Values

Ch 3: #38) Let $T_1$ and $T_2$ be independent exponentials with parameters $\lambda_1$ and $\lambda_2$. Find the density function of $T_1$ and $T_2$. 
Problem 2) Let \( X \) and \( Y \) be independent uniform \([0,1]\) random variables.

Consider the (seemingly ugly) transformations:

\[
U = \sqrt{-2 \ln(X)} \cos(2\pi Y) \\
V = \sqrt{-2 \ln(X)} \sin(2\pi Y)
\]

a) Demonstrate that:

\[
X = \exp\left(-\frac{U^2 + V^2}{2}\right) \\
Y = \frac{1}{2\pi} \arctan \frac{V}{U}
\]

b) Use the transformation of variable formula to find the joint distribution of \( U \) and \( V \), and remember to specify where it is defined.

c) Identify the joint distribution by name.
3.7 – Order Statistics (cont.)

Let $X_1, X_2, \ldots, X_n$ be independent random variables with the same CDF $F_X(x)$.

The values in order from lowest to smallest are the order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$.

The marginal p.d.f. for any of the order statistics is:

$$f_{X_{(k)}}(x_{(k)}) = \frac{n!}{(k-1)!(n-k)!} f_X(x_{(k)})$$

$$\cdot F_{k-1}(x_{(k)})[1 - F_X(x_{(k)})]^{n-k}$$

The joint p.d.f. of all of the order statistics is:

$$f_{X_{(1)}, \ldots, X_{(n)}}(x_{(1)}, \ldots, x_{(n)}) = n! f(x_{(1)}) \cdots f(x_{(n)})$$
One way to find the joint p.d.f. of a pair of order statistics would be to integrate out the \( n-2 \) you are not concerned with.

Another way is to use what the text calls “a differential argument” (Theorem A on 101 uses this to prove the result in the hmwk.)

Say we want the joint p.d.f. of \( X_{(i)} \) and \( X_{(j)} \) where \( i < j \).

The trick to getting the joint p.d.f. directly is try to let our insights into discrete distributions apply to continuous random variables.

In particular we will imagine that:

\[
f(x,y) = P[x ≤ X ≤ x + dx, y ≤ Y ≤ y + dy]
\]

\[
f(x) = P[x ≤ X ≤ x + dx]
\]

And so...

\[
f_{X_{(i)},X_{(j)}}(x_{(i)},x_{(j)}) = \\
\frac{n!}{(i-1)!(j-i-1)!(n-j)!} \cdot F_{X}^{i-1}(x_{(i)}) \cdot \left[ F_{X}(x_{(j)}) - F_{X}(x_{(i)}) \right]^{j-i-1} \cdot \left[ 1 - F_{X}(x_{(j)}) \right]^{n-j} \cdot f_{X}(x_{(i)}) f_{X}(x_{(j)})
\]
Chapter 4 Revisited: More on Expected Values

Recall that

\[ E(X) = \sum x p(x) \quad \Rightarrow \quad \int_{-\infty}^{+\infty} x f(x) dx \]
\[ Var(X) = \sum (x - \mu)^2 p(x) \quad \Rightarrow \quad \int_{-\infty}^{+\infty} (x - \mu)^2 f(x) dx \]

For constants a and b,

\[ E(a + bX) = a + b E(X) \]
\[ Var(a + bX) = b^2 Var(X) \]

Let \( X_1, X_2, \ldots X_n \) be mutually independent random variables, then:

\[ \mu_{\Sigma X} = E(\Sigma X_i) = \Sigma_i E(X_i) = \Sigma_i \mu_i \]
\[ \sigma_{\Sigma X}^2 = Var(\Sigma X_i) = \Sigma_i Var(X_i) = \Sigma \sigma_i^2 \]
What if the $X_i$ are not independent?

First, if the $X_i$ have joint p.d.f $f(x_1,\ldots,x_n)$ and $Y=g(x_1,\ldots,x_n)$ then

$$E(Y) = \int \cdots \int g(x_1,\ldots,x_n) f(x_1,\ldots,x_n) dx_1 \cdots dx_n$$

Provided the integral converges with $|g|$.

Now consider $Y = a + b \sum_{i=1}^{n} X_i$

and finding $E(Y)$ and $\text{Var}(Y)$.