Today

• Order Statistics

3.7 – Order Statistics

Let $X_1, X_2, \ldots, X_n$ be independent random variables with the same CDF $F_X(x)$.

The values in order from lowest to smallest are the order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$. 
First consider the maximum \( U=X_{(n)} \).
Note that \( U \leq u \) if and only if all of the \( X_i \leq u \).

\[
F_U(u) = P(U \leq u) \\
= P((X_1 \leq u) \cap \cdots \cap (X_n \leq u)) \\
= P(X_1 \leq u) \cdots P(X_n \leq u)
\]

Taking the derivative we get:

\[
f_U(u) = n f_X(u) [F_X(u)]^{n-1}
\]

The minimum \( V=X_{(1)} \) works similarly:

\[
F_V(v) = 1 - [1 - F_X(v)]^n \\
f_V(v) = n f_X(v) [1 - F_X(v)]^{n-1}
\]
This method is a bit messier to use for the other order statistics. Another option is the “differential argument”.

The trick to getting the joint p.d.f. directly is try to let our insights into discrete distributions apply to continuous random variables.

And we get...

\[ f_{X(k)}(x_{(k)}) = \frac{n!}{(k-1)!(n-k)!} f_X(x_{(k)}) \cdot F^{k-1}(x_{(k)})[1 - F_X(x_{(k)})]^{n-k} \]

Example 1) Say you conduct 10 independent tests of hypotheses. How small should the smallest p-value be for you to reject it at a 0.05 level?

That is, what is the 5th-%ile for the 1st order statistic?
Example 2) $x_{(k)}$ for a uniform random variable.

This is a beta distribution with parameters $k$ and $n-k+1$. 
The joint p.d.f. of all of the order statistics is:

\[ f_{X(1), \ldots, X(n)}(x_{(1)}, \ldots, x_{(n)}) = \]

\[ n! f(x_{(1)}) \cdots f(x_{(n)}) \]

One way to find the joint p.d.f. of a pair of order statistics would be to integrate out the \( n-2 \) you are not concerned with.

Another way is to use what the “differential argument”.

Say we want the joint p.d.f. of \( X_{(i)} \) and \( X_{(j)} \) where \( i < j \).

And so...

\[ f_{X(i), X(j)}(x_{(i)}, x_{(j)}) = \]

\[ \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \cdot F_X^{i-1}(x_{(i)}) \cdot [F_X(x_{(j)}) - F_X(x_{(i)})]^{j-i-1} \cdot [1 - F_X(x_{(j)})]^{n-j} \cdot f_X(x_{(i)}) f_X(x_{(j)}) \]