Today

• Exam 2

• Moment Generating Functions continued

Exam 2 #1) Let X and Y be independent random variables where X is exponential with λ=2 and Y is normal with μ=4 and σ=6. Find E(X+Y) and Var(X+Y).
2) Find k such that \( f(x) = kx(1-x) \) on \( 0 < x < 1 \) (and is 0 elsewhere) is a p.d.f.

3) Let the random variable \( X \) have c.d.f. \( F(x) = \frac{1}{2}(1 + x^3) \) on \(-1 \leq x \leq 1\) (and 0 otherwise). Find \( E(X) \) and \( \text{Var}(X) \).

4) Leaks due to manufacturing defects occur in a brand of hose at a rate of approximately 1 per 500 feet. Name an appropriate distribution and estimate the probability that the first defect will be found in the first 100 feet.
5) Evaluate \( \int_{0}^{\infty} x^2 e^{-\pi x} \, dx \)

6) Let \( X \) have a uniform distribution on \((-\pi/2, \pi/2)\). Find the c.d.f. and p.d.f. of \( Y = \tan X \).

7) Let \( f_{X,Y}(x,y) = \frac{1}{4 + xy} \) on \(-1 < x < 1, -1 < y < 1\) (and be 0 otherwise). Find the conditional distributions of \( X|Y \) and \( Y|X \). Also, are \( X \) and \( Y \) independent?
8) Let $X$ and $Y$ be independent chi-square random variables with 1 degree of freedom. (The p.d.f. is on page 59.) Derive the p.d.f. of $Z = X/Y$.

9) In class we showed how to get general formula's for the p.d.f.'s of $Z = X/Y$ and $Z = X + Y$. Show that if $X$ and $Y$ have joint p.d.f. $f_{XY}(x,y)$ and $Z = XY$ that

$$\frac{f_Z(z)}{z} = \int f_{XY}(x, \frac{z}{x}) \left( \frac{1}{x} \right) dx.$$ 

10) Let $X$ and $Y$ have joint p.d.f. $f_{XY}(x,y) = 1 + (1-2x)(1-2y)$ on $0 < x < 1, 0 < y < 1$ (and be 0 elsewhere). Find the joint p.d.f. of $U = X + Y$ and $V = X + 2Y$. 
11) Let $X_1, \ldots, X_{11}$ be independent exponential random variables with parameter $\lambda = 1$. Find the p.d.f. for the median $X_{(6)}$.

4.5 – Moment Generating Functions

The moment-generating function (mgf) of $X$ is $M(t) = E(e^{tX})$

$$M(t) = \sum_{x} e^{tx} p(x)$$

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

Properties of m.g.f.'s

a) If the m.g.f. exists on an interval around zero then $M^{(k)}(0) = E(X^k)$

b) The m.g.f. uniquely determines the p.d.f.

c) If $Y = a + bX$ then $M_Y(t) = e^{at} M_X(bt)$

d) If $X$ and $Y$ are independent and $Z = X + Y$ then $M_Z(t) = M_X(t) M_Y(t)$
Example 1) \( X \sim \text{Uniform}[0,1] \)

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M_X(t) = \]

\[
M_{aX+b}(t) =
\]

Example 2) Sum of Negative Binomials.

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M(t) = \frac{\left(pe^t\right)^r}{\left[1-(1-p)e^t\right]^r} \quad \text{for} \quad t < -\ln(1-p)
\]