1) Consider a random sample \(x_1, \ldots, x_n\) from a distribution with pdf
\[
f(x) = (\theta + 1)(1-x)^\theta \quad \text{for} \ 0 < x < 1
\]

a) Find the MoM estimator for \(\theta\).

\[
\mu = \int x(\theta + 1)(1-x)^\theta \, dx \quad \text{Let } y=1-x, \text{ so that } x=1-y, \text{ and } dx=-dy
\]

\[
= -\int_{x=0}^{x=1} (1-y)(\theta + 1)y^\theta \, dy = -(\theta + 1) \int_{x=0}^{y=1} (y^\theta - y^{\theta+1}) \, dy = -(\theta + 1)(\frac{y^{\theta+1}}{\theta + 1} - \frac{y^{\theta+2}}{\theta + 2})_{y=0}^{y=1} = (\theta + 1)(\frac{1}{\theta + 1} - \frac{1}{\theta + 2}) = \frac{1}{(\theta + 2)}
\]

Or, notice that this is a beta distribution with \(\alpha=1\) and \(\beta=\theta+1\).

So \(\bar{x} = \frac{1}{\hat{\theta}_{mom} + 2} \Rightarrow \hat{\theta}_{mom} + 2 = \frac{1}{\bar{x}} \Rightarrow \hat{\theta}_{mom} = \frac{1}{\bar{x}} - 2\)

b) Find the MLE estimator for \(\theta\).

\[
\text{lik}(\theta | x_1, \ldots, x_n) = f(x_1, \ldots, x_n | \theta) = \prod_{i=1}^{n} f(x_i | \theta) = \prod_{i=1}^{n} (\theta + 1)(1-x_i)^\theta
\]

\[
\log \text{lik}(\theta | x_1, \ldots, x_n) = \sum_{i=1}^{n} [\log(\theta + 1) + \theta \log(1-x_i)] = n \log(\theta + 1) + \theta \sum_{i=1}^{n} \log(1-x_i)
\]

\[
\frac{\partial}{\partial \theta} \log \text{lik}(\theta | x_1, \ldots, x_n) = \frac{n}{\theta + 1} + \sum_{i=1}^{n} \log(1-x_i) = 0
\]

\(\Rightarrow \hat{\theta}_{mle} = \frac{n}{\sum_{i=1}^{n} \log(1-x_i)} - 1\)
2) In many cases a process will have a minimum value > 0 so that using a distribution like the exponential, chi-squared, F, or gamma doesn’t make much sense. In this case the distribution can be shifted to the right. Consider the shifted exponential with pdf

\[ f(x) = \frac{1}{\theta} e^{-\frac{(x-a)}{\theta}} \text{ for } x>a \]

and the data set 16.2, 12.4, 6.0, 8.4, 6.8, 9.1, 6.6, 6.0, 10.7, 5.8.

a) Show that the mean of the shifted exponential is \( \theta + a \) and the variance is \( \theta^2 \).

\[
\mu = \int_{a}^{\infty} x f(x) dx = \int_{a}^{\infty} \frac{x}{\theta} e^{-\frac{(x-a)}{\theta}} dx \quad \text{let } y=x-a \text{ so that } x=y+a \text{ and } dx=dy \text{ and we get}
\]

\[
= \int_{x=a}^{\infty} (y+a) \frac{1}{\theta} e^{-\frac{y}{\theta}} dy = a + \int_{y=0}^{\infty} y \frac{1}{\theta} e^{-\frac{y}{\theta}} dy = a + \theta \text{ because } y \text{ is just exponential.}
\]

\[
\sigma^2 = \int_{a}^{\infty} (x-(\theta + a))^2 f(x) dx = \int_{a}^{\infty} (x+\theta - (\theta + a))^2 \frac{1}{\theta} e^{-\frac{(x-a)}{\theta}} dx \text{ again let } y=x-a
\]

\[
= \int_{x=a}^{\infty} (y-\theta)^2 \frac{1}{\theta} e^{-\frac{y}{\theta}} dy = \theta^2 \text{ because } y \text{ is again just exponential.}
\]

b) Find the form of the MoM estimators for \( a \) and \( \theta \).

\[
\bar{x} = \hat{a}_{mom} + \hat{\theta}_{mom}
\]

\[
\hat{\sigma}^2 = \hat{\theta}_{mom}^2 \Rightarrow \hat{\theta}_{mom} = \hat{\sigma} \Rightarrow \hat{a}_{mom} = \bar{x} - \hat{\sigma}
\]

In this case we get \( \hat{\theta}_{mom} = \hat{\sigma} = 3.242 \) and \( \hat{a}_{mom} = \bar{x} - \hat{\sigma} = 8.8 - 3.242 = 5.558 \)

c) Find the form of the MLE for \( a \) and \( \theta \). (Note that when you take the derivative with respect to \( a \) that it can never equal zero, so the maximum must happen at one of the end-points. Also note that the bigger \( a \) is the bigger the log-likelihood is. Based on what you know about the pdf, what is the biggest value that \( a \) can have?)

\[
lik(a, \theta \mid x_1, \ldots, x_n) = f(x_1, \ldots, x_n \mid \theta) = \prod_{i=1}^{n} f(x_i \mid \theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{(x_i-a)}{\theta}}
\]

\[
\log lik(a, \theta \mid x_1, \ldots, x_n) = \sum_{i=1}^{n} [-\log \theta - \frac{(x_i-a)}{\theta}] = -n \log \theta - \frac{n}{\theta} \sum_{i=1}^{n} x_i + \frac{na}{\theta}
\]

\[
\frac{\partial}{\partial a} \log lik(a, \theta \mid x_1, \ldots, x_n) = \frac{n}{\theta} \text{ which is never 0! So must be a boundary. The biggest it can be is the smallest observed } x!
\]

\[
\frac{\partial}{\partial \theta} \log lik(a, \theta \mid x_1, \ldots, x_n) = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} x_i}{\theta^2} - \frac{na}{\theta^2} = 0 \Rightarrow n \theta = \sum_{i=1}^{n} x_i - na \Rightarrow \theta = \bar{x} - a
\]

So we get that \( \hat{a}_{mle} = x_{(1)} \) and \( \hat{\theta}_{mle} = \bar{x} - x_{(1)} \) which in this case are \( \hat{a}_{mle} = 5.8 \) and \( \hat{\theta}_{mle} = 8.8 - 5.8 = 3 \)
3) Consider the Cauchy distribution centered at $\theta$, that has pdf

$$f(x) = \frac{1}{\pi(1 + (x - \theta)^2)}$$

for $-\infty < x < \infty$

a) Why can’t there be a MoM estimator for $\theta$?

The Cauchy distribution doesn’t have any moments!

b) What formula must the MLE satisfy?

$$lik(\theta | x_1, ..., x_n) = f(x_1, ..., x_n | \theta) = \prod_{i=1}^{n} f(x_i | \theta) = \prod_{i=1}^{n} \frac{1}{\pi(1 + (x_i - \theta)^2)}$$

$$\log lik(\theta | x_1, ..., x_n) = \sum_{i=1}^{n} \left[ -\log \pi - \log(1 + (x_i - \theta)^2) \right] = -n \log \pi - \sum_{i=1}^{n} \log(1 + (x_i - \theta)^2)$$

$$\frac{\partial}{\partial \theta} \log lik(\theta | x_1, ..., x_n) = -\sum_{i=1}^{n} \frac{2(x_i - \theta)}{1 + (x_i - \theta)^2} = 0 \Rightarrow \hat{\theta}_{mle} \text{ must solve } \sum_{i=1}^{n} \frac{(x_i - \theta)}{1 + (x_i - \theta)^2} = 0$$

c) Use R to find the estimate of $\theta$ based on the sample –0.6, 4.2, 1.1, -4.3, -10.3, 1.6, 4.8, 30.9, 0.4, 1.5.

```r
x<-c(-0.6, 4.2, 1.1, -4.3, -10.3, 1.6, 4.8, 30.9, 0.4, 1.5)

cauchynloglik<-function(theta, data){
  n<-length(data)
  x<-data
  -(n*log(pi)-sum(log(1+(x-theta)^2)))
}

optim(0,cauchynloglik,method="BFGS",data=x)

$par
[1] 1.171398
```