Today

Some Properties of Estimators

- Efficiency
- Cramer-Rao Inequality
- Sufficiency

8.7 Comparing Estimates and Tests

One of the standard tools for evaluating an estimate is the mean squared error:

\[ MSE(\hat{\theta}) = E(\hat{\theta} - \theta_0)^2 = Var(\hat{\theta}) + [E(\hat{\theta}) - \theta_0]^2 \]
If two estimators are unbiased, then the efficiency of $\hat{\theta}$ relative to $\tilde{\theta}$ is

$$
\text{eff}(\hat{\theta}, \tilde{\theta}) = \frac{\text{var}(\tilde{\theta})}{\text{var}(\hat{\theta})}
$$

For two tests $T_1$ and $T_2$ of the same $H_0$ and $H_A$ with the same $\alpha$-level, the relative efficiency of $T_1$ to $T_1$ is the ratio $n_2/n_1$ required so that they have the same power.

The asymptotic relative efficiency of the MWW to the $t$-test is:
- 0.955 if the populations are normal
- 1.0 if the populations are uniform
- 1.5 if the pops. are double-exp.
- 0.864 to infinity in general

assuming the populations differ only by location.
The asymptotic relative efficiency of the MWW to the median test is:
• 1.5 if the populations are normal
• 3.0 if the populations are uniform
• 0.75 if the pops. are double-exp.
assuming the populations differ only by location.

Cramer-Rao Inequality
Let $X_1, \ldots, X_n$ be i.i.d. with density $f(x | \theta)$, and $T$ be an unbiased estimate of $\theta$. Then under appropriate smoothness assumptions on $f$
$$Var(T) \geq \frac{1}{nI(\theta)}$$

Recall that under smoothness conditions that the $1/n I(\theta)$ is the asymptotic variance of the MLE!

So, why isn’t the MLE always best?
8.8 Sufficiency  One of the key concepts in advanced mathematical statistics is that of sufficiency. Does a statistic summarize all of the information in the data about a parameter, or do we lose something by summarizing.

Defn  A statistic $T(X_1, \ldots X_n)$ is sufficient for $\theta$ if the conditional distribution of $X_1, \ldots X_n$ given $T=t$ does not depend on $\theta$ for any value of $t$.

Example: Consider a Poisson Distribution with parameter $\lambda$ and a sample size of 2.

A) Consider $T=X_1 + X_2$

B) Consider $T=X_1 + 2X_2$