Today

Section 8.4: Method of Moments
- Estimating $\alpha$ and $\lambda$ for the Gamma
- Q-Q plots
- The Parametric Bootstrap
- Consistency

A sampling distribution is the probability distribution of a statistic.

In general we want a sampling distribution that is as close as possible to the corresponding parameter.
8.4 – Method of Moments (cont.)

Use a number of moments equal to the number of parameters that need to be estimated, and set the sample moments equal to the distribution’s moments.

Recall that...

\( \mu_1 = E(X) = \mu \)
\( \mu_2 = E(X^2) = \text{Var}(X) + (E(X))^2 = \sigma^2 + \mu^2 \)
\[
\hat{\mu}_1 = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{X} \quad \hat{\mu}_2 = \frac{\sum_{i=1}^{n} x_i^2}{n} = \frac{\sum_{i=1}^{n} (x_i - \bar{X})^2}{n} = \frac{\sum (x_i - \bar{X})^2 + n\bar{X}^2}{n} = \delta^2 + \bar{X}^2
\]

Consider a Gamma distribution where:

\( \mu = \frac{\alpha}{\lambda} \)
\( \sigma^2 = \frac{\alpha}{\lambda^2} \)
Example: Discrimination parameters for a law school admissions test.
0.52208 0.61226 0.61651
0.67259 0.68124 0.70027
0.79531 0.80179 0.85638
0.87090 0.88407 0.90651
0.95291 0.99212 1.08418
1.09365 1.23861 1.36625
1.36719 1.57871 1.61840
1.67781 1.77927 2.02504

> hist(a)

> mean(a)
[1] 1.070585

> (n-1)/n*var(a)
[1] 0.1691372

Question 1: Does the gamma with these parameters seem to match our data?

A quantile-quantile plot of our data against $F^{-1}(i/(n+1))$ could be used to see.
\begin{verbatim}
xbar<-mean(a)
sigma2hat<-(n-1)/n*var(a)
lhat<-xbar/sigma2hat
ahat<-xbar^2/sigma2hat
n<-length(a)

plot(sort(a),qgamma((1:n)/(n+1),shape=ahat,rate=lhat))
lines(qgamma((1:n)/(n+1),shape=ahat,rate=lhat),qgamma((1:n)/(n+1),shape=ahat,rate=lhat))
\end{verbatim}

Say we repeated the plot with
\begin{verbatim}
a<-rnorm(24,1.07,sqrt(.1677))
\end{verbatim}
Question 2: How accurate are the estimates?

If we had the actual $\alpha$ and $\lambda$, we could get a large number of samples of size 24 from that distribution and calculate the estimates for each one.

Since we don't have the true $\alpha$ and $\lambda$, the best we can do is to use the estimates instead.

This is called a parametric bootstrap.

```r
nsamples<-100000
x<-rgamma(n*nsamples,shape=ahat, rate=lhat)
x<-matrix(x,ncol=n)
xbar.dist<-apply(x,1,mean)
s2h.dist<-(n-1)/n*apply(x,1,var)
lhat.dist<-xbar.dist/s2h.dist
ahat.dist<-xbar.dist^2/s2h.dist
```
Unfortunately we generally have no way of knowing exactly how well the method of moment estimators will behave in general.

We do, however, know that they are consistent.

**Definition:** Let $\hat{\theta}_n$ be an estimate of $\theta$ for a sample of size $n$. $\hat{\theta}_n$ is said to be a consistent estimator of $\theta$ if it converges to $\theta$ in probability.

That is, if for any $\varepsilon>0$,

$$P(|\hat{\theta}_n - \theta| > \varepsilon) \to 0 \text{ as } n \to \infty$$