STAT 703/J703
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-*Lecture 9*-

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Today

- Asymptotic Normality of the MLE (continued)
- Basics of Hypothesis Testing (Chapter 9)



Theorem B: Under appropriate regularity conditions the MLE is asymptotically normal with mean $\theta$ and variance $\dfrac{1}{nI(\theta)}$ .

Sketch of proof: Consider the Taylor series expansion:

$$L'(\hat{\theta}) \approx L'(\hat{\theta}) + (\hat{\theta} - \theta)L''(\hat{\theta})$$



Statistical Inference – Confidence Intervals and Tests of Hypotheses

Test of hypothesis – general method to distinguish between 2 (or more) probability distributions (or models), based on a sample $X_1, \ldots, X_n$ assumed to come from one of them.



In particular, based on $X_1, \ldots, X_n$, decide whether $f_1(x)$ or $f_2(x)$ is the pdf (or population) from which the sample came.

More specifically, suppose we think the sample is from a normal population with mean either $\mu=5$ or $\mu=10$ with variance 4. (Or, more generally, $\mu=\mu_1$ vs. $\mu=\mu_2$).

To do this, we follow the "Neyman-Pearson Paradigm" and discuss "significance tests".

Neyman-Pearson Approach:

Group the possible hypothesized distributions into two categories, the null hypothesis, $H_0$, and the alternative hypothesis, $H_A$ (research hypothesis).



E.g. Observe $X_1, \ldots, X_n$.

Either $H_0$: $N(\mu_1, \sigma^2)$ or $H_A$: $N(\mu_2, \sigma^2)$

or $H_0$: $\mu=\mu_1$ vs. $H_A$: $\mu=\mu_2$,

$\mu$ is the mean of $N(\mu, \sigma^2)$.

Here, if $\sigma^2$ is known, each of these hypotheses completely specifies the distribution or population. So, $H_0$ and $H_A$ are called simple hypotheses.



If $H_0$: $\mu=0$ vs. $H_A$: $\mu > 0$ ($\mu$ in $N(\mu, \sigma^2)$, with $\sigma^2$ known).

Then $H_0$ is simple and $H_A$ is a composite hypothesis, i.e. several normal distributions would satisfy it.

$H_A$ is also referred to as a one-sided hypothesis.

If $H_0$: $\mu$=0 vs. $H_A$: $\mu \neq 0$ ($\mu$ in N($\mu$, $\sigma^2$), $\sigma^2$ known), then $H_A$ is a <u>two-sided</u> (composite) hypothesis.

Next, we set up the framework for "testing" $H_0$ and $H_A$ based on the sample.



<u>9.2 Neyman-Pearson Paradigm</u>

Let $\underline{X} = (X_1, \ldots, X_n)$ denote a sample from population f(x|θ). Decide on $H_0$ vs. $H_A$ based on the sample.

A decision on whether or not to reject $H_0$ in favor of $H_A$ is made on the basis of a statistic

$$T=T(\underline{X})=T(X_1, \ldots, X_n).$$



The set of values of T for which $H_0$ is accepted is called the <u>acceptance region</u> and the set of values of T for which $H_0$ is rejected is the <u>rejection region of the test</u>.

Two kinds of error may occur:

1. $H_0$ is rejected when it is true: Type I error.

P(type I error) = $\alpha$
= P(T $\in$ rejection region | $H_0$ true).

If $H_0$ is simple, $\alpha$ is called the significance level of the test.

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2. $H_0$ is accepted when it is false: Type II error.

P(type II error) = $\beta$
= P(T in acceptance region | $H_0$ false)

If $H_A$ is composite, $\beta$ depends on which member of $H_A$ is the true pdf.

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Power of the test =P($H_0$ is rejected when false)
= 1 - P($H_0$ is accepted | $H_0$ false)
= 1 - $\beta$.

Ideally, we would want $\alpha$=$\beta$=0, but this not possible since the decision is based on data.

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Example 1: Consider a sample of size 1 from a normal distribution with variance 1.

Test $H_0$:$\mu$=0 vs. $H_A$:$\mu$=1 at $\alpha$=0.05.



Example 2: Consider a sample of size 1 from a normal distribution with variance 1.

Test $H_0$:$\mu\leq$0 vs. $H_A$:$\mu$>0 at $\alpha$=0.05.



For a composite test the significance level $\alpha$ is the maximum (supremum) of the probabilities of a Type I error over all the possible alternatives.