Today

Properties of Estimators cont.

- Sufficiency
- Factorization Theorem
- Rao-Blackwell
- And Beyond!

8.7 Sufficiency One of the key concepts in advanced mathematical statistics is that of sufficiency. Does a statistic summarize all of the information in the data about a parameter, or do we lose something by summarizing.
**Defn** A statistic $T(X_1, \ldots X_n)$ is sufficient for $\theta$ if the conditional distribution of $X_1, \ldots X_n$ given $T=t$ does not depend on $\theta$ for any value of $t$.

**Example 1:** Poisson Distribution

**Example 2:** Normal Distribution
The Factorization Theorem
A necessary and sufficient condition for $T$ to be sufficient for $\theta$ is that the joint p.d.f. factors in the form:

$$f(x_1, \ldots, x_n \mid \theta) = g(T(x_1, \ldots, x_n), \theta)h(x_1, \ldots, x_n)$$

Rao-Blackwell Theorem
Let $\hat{\theta}$ be an estimator of $\theta$ with $E(\hat{\theta})^2 < \infty$ for all $\theta$. If $T$ is sufficient for $q$ then $\bar{\theta} = E(\hat{\theta} \mid T)$ satisfies

$$E(\bar{\theta} - \theta)^2 \leq E(\hat{\theta} - \theta)^2$$

Example 3: Consider trying to estimate $\lambda$ for a Poisson distribution using only $X_1$. 
Lehmann-Scheffe Theorem
Let \( \hat{\theta} \) be an unbiased estimator of \( \theta \) with \( \mathbb{E}(\hat{\theta})^2 < \infty \) for all \( \theta \). If \( T \) is complete sufficient for \( \theta \) then \( \theta = E(\hat{\theta} \mid T) \) is the uniformly minimum variance unbiased estimate (UMVUE) of \( \theta \).

Complete? A statistic \( T \) is complete for \( \theta \) if the zero function is the only function that satisfies:
\[
E_\theta[g(T)] = 0 \text{ for all } \theta
\]

However we have a result similar to the factorization theorem for “exponential families”.
\[
f(x \mid \theta) = \exp\left[\sum_{i=1}^{k} T_i(x)c_i(\theta) + d(\theta) + S(x)\right]
\]
where \( \theta = (\theta_1, \ldots, \theta_k) \)
Under appropriate regularity conditions the vector of \( T \)'s is complete sufficient.