2.2

Unions and intersections

$\boxed{\text{Null Event}}$

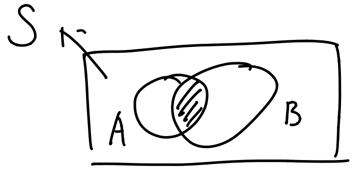
The null event, denoted by ϕ is an event that contains no outcomes. The null event event has probability $P(\phi) = 0$.

$\boxed{\text{Union}}$ The union of two events A and B, contains all outcomes w in either event or both. We denote the union of two events A and B by $A \cup B = \{w : w \in A \text{ or } w \in B\}$.

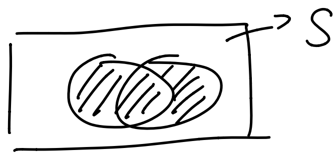
$\boxed{\text{Intersection}}$ The intersection of two events A and B contains all outcomes w in both events. We denote the intersection of two events A and B by $A \cap B = \{w : w \in A \text{ and } w \in B\}$.

$\boxed{\text{Mutually Exclusive}}$ If A and B contains no common outcomes, we say the events are mutually exclusive.

In other words $P(A \cap B) = P(\phi) = 0$.

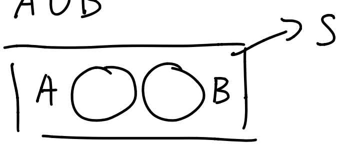


$A \cap B$



$A \cup B$

If A & B are disjoint



"I'm emphasizing S Because A & B are events. And thus they are all subsets of S.".

Example 2.3

Help understanding the intersection and union.

Hemophilia is a sex-linked hereditary blood defect of males. (Delayed clotting of blood) When a woman is a carrier of classical hemophilia, there is 50% chance that a male child will inherit this disease. If the carrier gives birth to two boys, what is the probability of either of them will have the disease? both will have the disease.?.

Solution:

We envision the process of having two male children as an experiment with sample space

$$S = \{++, +-, -+, --\}.$$

[Note that "++" is an outcome, but + is not an outcome].

[flip a coin for 3 times. what is the sample space].

[where "+" means male off-springs has the disease and "–" means male offspring doesn't have the disease].

$$A = \{\text{first child has disease}\} = \{++, +-\}.$$

$$B = \{\text{second child has disease}\} = \{++, -+\}.$$

[curly bracket, for events. it's not arbitrary].

The union and intersection of A and B are

$$A \cup B = \{\text{either child has disease}\}.$$

$$= \{++, +-, -+\} \text{ [Non-repetitive]}.$$

$A \cap B = \{$ both children has disease $\} = \{++\}$.

The probability of either male child will have the disease is $P(A \cup B) = \frac{n_{A \cup B}}{n_S} = 3/4 = 0.75$.

[to compute the probabilities, we'll assume that each outcome in S is equally likely].

The probability of both male child will have the disease is $P(A \cap B) = \frac{n_{A \cap B}}{n_S} = \frac{1}{4} = 0.25$.

2.3 axioms of probability.

not gonna prove it.

Kolmogorovs Axioms: For any sample space S, a probability P must satisfy

[like when you assign probs to an event].

(1) $0 \leq P(A) \leq 1$ for any event A.

(2) $P(S) = 1$ "S is the sample space"

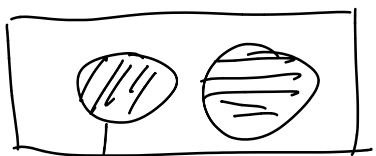
Start from side note, which is easien to understand.

(3) If $A_1, A_2, A_3 \ldots A_n$ are pairwise mutually

Exclusive events, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

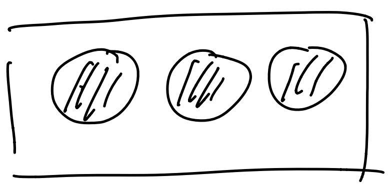
$A_i, \ A_j$



$A_i \quad A_j$

$$P(\cancel{A_i \cap A_j}\ A_i \cup A_j) = P(A_i) + P(A_j)$$

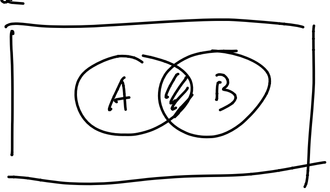
$\rightarrow A_i, A_j, A_k$



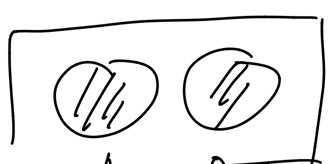
$$P(A_1 \cup A_2 \cup A_3) = \sum_{i=1}^{3} P(A_i) = P(A_1) + P(A_2) + P(A_3)$$

$\boxed{\text{Start from here}}$.

Side note



$$\cancel{P(A)}\ P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



$A \quad B$

$$P(A \cup B) = P(A) + P(B)$$

2.4. Conditional probability and independence

~~Note~~: $P(T) = 0.50.$

$P(T \mid \text{college graduate}) = 0.42.$

$P(T \mid \text{evangelical}) = 0.80.$

$P(T \mid 18\text{-}29) = 0.36.$

CNN Exit polls.

Note: In some situations, we may have prior knowledge about the likelihood of other events related to the event of interest. We can then incorporate this information into a probability calculation.

$\boxed{\text{Conditional probability}}$.

Let A, B be the events in a sample space S with $P(B) > 0$, The conditional probability of A, given that B has occurred, is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Similarly $P(B|A) = \frac{P(A \cap B)}{P(A)}$. [NOT interchangable].

Example 2.4. SEC

In a company, 36 percent of the employees have a degree from a SEC university

22% of those employees with an SEC degree are engineers.

And 30% of employee's are engineers.

An employee is selected at random.

(a) Compute the probability of the employee is an engineer and is from the SEC

(b). Compute the conditional probability that the employee is from SEC Given that he/she is an engineer.

Solution:

Define the events.

$A = \{$ employee is an engineer $\}$.

$B = \{$ employee is from the SEC $\}$

From the information in the problem, we are given

$P(A) = 0.30$

$P(B) = 0.36$

$P(\cancel{B|A}) = (A|B) = 0.22$.

In part A, ~~Defin~~ what we want is $P(A \cap B)$

Note that

$$0.22 = P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = 0.22 \times P(B) = 0.22 \times 0.36 = 0.0792$$

In part (B), we want $P(B|A)$. From the definition, we only need

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.0792}{0.30} = 0.264.$$

FACT One: $P(A|B) \neq P(B|A)$

$P(A|B) = 0.22$.

$P(B|A) = 0.264$

FACT TWO: $P(B|A) \neq P(B)$

↓ conditional probability

↓ unconditional probability.

"In other words, knowledge that A has occurred" has changed the likelihood that B occurs".

⇓.

What if occurrence of B has no effect on the probability of A.

Then we got A and B are independent.

Independent

When the occurrence of B (or non-occurrence) has no effect on whether or not A occurs, and vice versa, we say that the events A & B are independent.

Mathematically, A and B are independent if and only if.

(3). $$P(A \cap B) = P(B) \cdot P(A).$$

Hence we can derive

(1) $$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

(2) $$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B).$$

(1) & (2) only hold when (3) holds.

Actaully as long as one of them holds, all others would hold.

Again, Revisit

$$P(B|A) \neq P(B)$$

0.264. v.s 0.36.

<u>And thus A & B are not independent.</u>

Check one of these, if asked about independence

E) $\boxed{\text{Mutual independence.}}$

The notion of independence can be extended to any finite collection of events $A_1 \cdots A_n$.

Mutual independence means that the probability of intersection of any sub-collection of $A_1 \cdots A_n$ equals the product of the probabilities of the events in the sub-collection.

For example, if A_1 A_2 A_3 and A_4 are mutually independent, then

$$P(A_1 \cap A_2) = P(A_1)P(A_2)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\,P(A_2)\,P(A_3)$$

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1)\,P(A_2)\,P(A_3)\,P(A_4).$$

2.5 probability rules.

(plus minus, multiply & division)

$\boxed{\text{Complement:}}$

Suppose S is a sample space and that A

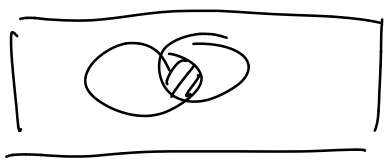
is an event, The complement of A, denoted by $\bar{A}$, is the collection of All outcomes in S not in A. That is

$$\bar{A} = \{ w \in S : w \notin A \}.$$

① Complement Rule

Suppose A is an event, $P(\bar{A}) = 1 - P(A)$

② Additive Law (suppose A & B are two events).



$$P(A \cup B) = P(A) + P(B) - \boxed{P(A \cap B)}$$ if mutually exclusive.

③ Multiplicative Law

Suppose A & B are two events.

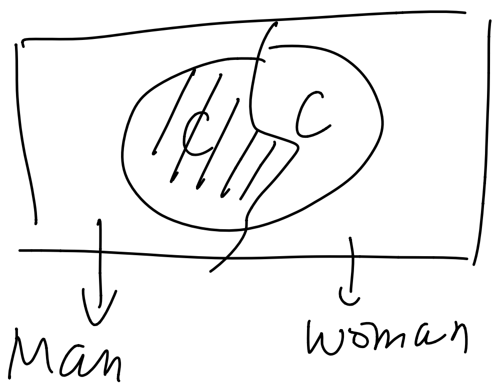
$$P(A \cap B) = P(A)\,P(B|A)$$
$$= P(A|B) \cdot P(B)$$

(If independent $P(A)P(B) = P(A \cap B)$.

4 Law of Total probability

suppose A & B are two events

$$P(A) = P(A|B)\,P(B) + P(A|\bar{B})\,P(\bar{B})$$



$\cancel{P(A|B)}$ $P(C) = P(\text{College and Man})$

$+ P(\text{College and woman})$

$= P(\text{college}|\text{Man}) \cdot P(\text{Man})$

$+ P(\text{College}|\text{Woman})\,P(\text{Woman})$

5 Baye' Rule

$$P(B|A) = \frac{P(A|B)\cdot P(B)}{P(A)} = \frac{P(A|B)\,P(B)}{P(A|B)\,P(B) + P(A|\bar{B})\,P(\bar{B})}$$

$P(B|A) \cdot P(A) = P(A|B) \cdot P(B)$

From $B|A$ we can infer $A|B$.

$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A)}$$

From Law of Total probability

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A|\bar{B})P(\bar{B}) + P(A|B)\,P(B)}$$

Example 2.6

~~Train~~ Probability of train 1 being on time 0.95

The probability of train 2 is on time is 0.93.

The probability that both are on time is 0.90.

$A_1 = \{\text{train 1 on time}\}$

$A_2 = \{\text{train 2 on time}\}$. We are given that

$$P(A_1) = 0.95.$$
$$P(A_2) = 0.93.$$
$$P(A_1 \cap A_2) = 0.90$$

① What's the probability of train 1 is not on time

$$P(\bar{A_1}) = 1 - P(A_1) = 1 - 0.95 = 0.05$$

② What's the probability that at least one train is ~~one~~ on time?

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2).$$

As long as one of the ~~the~~ two happens.

$$= 0.95 + 0.93 - 0.90 = 0.98$$

(c) Give that train 2 is on time what is the probability of train 1 being on time

$$P(A_1 \mid A_2) = \frac{P(A_1 \cap A_2)}{P(A_2)} = \frac{0.90}{0.93}$$

$$= 0.968$$

(d) What is the probability that train 2 is on time, given that train one is not on time.

$$P(A_2 \cap \bar{A}_1) + P(A_2 \cap A_1) = P(A_2)$$

$$P(A_2 \mid \bar{A}_1) = \frac{P(A_2 \cap \bar{A}_1)}{P(\bar{A}_1)}$$

$$= \frac{P(A_2) - P(A_1 \cap A_2)}{1 - P(A_1)}$$

$$= \frac{0.93 - 0.90}{1 - 0.95} = \frac{0.03}{0.05} = 0.60$$

(e) are A_1 & A_2 independent from each other.

$$0.968$$

$$P(A_1|A_2) = P(A_1) \quad 0.95$$

Hence A_1 & A_2 are not independent.