Thursday, July 7, 2016

Homework 2

STAT 512: Mathematical Statistics

Deadline: July, 11TH, Before Class

Question 1

A random variable Y has a *beta distribution of the second kind*, if, for $\alpha > 0$ and $\beta > 0$, its density is

$$f_Y(y) = \begin{cases} \dfrac{y^{\alpha-1}}{B(\alpha,\beta)(1+y)^{\alpha+\beta}}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Derive the density function of $U = 1/(1+Y)$.

Question 2

If Y is a continuous random variable with distribution function $F(y)$, find the probability density function of $U = F(Y)$.

Question 3

If a random variable U is normally distributed with mean μ and variance σ^2 and $Y = e^U$ [equivalently, $U = \ln(Y)$], then Y is said to have a *log-normal distribution*. The log-normal distribution is often used in the biological and physical sciences to model sizes, by volume or weight, of various quantities, such as crushed coal particles, bacteria colonies, and individual animals. Let U and Y be as stated. Show that

a the density function for Y is

$$f(y) = \begin{cases} \left(\frac{1}{y\sigma\sqrt{2\pi}}\right) e^{-(\ln y - \mu)^2/(2\sigma^2)}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

b $E(Y) = e^{\mu+(\sigma^2/2)}$ and $V(Y) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$. [*Hint:* Recall that $E(Y) = E(e^U)$ and $E(Y^2) = E(e^{2U})$, where U is normally distributed with mean μ and variance σ^2. Recall that the moment-generating function of U is $m_U(t) = e^{tU}$.]

Question 4

Suppose that Y_1 has a gamma distribution with parameters α_1 and β, that Y_1 is gamma distributed with parameters α_2 and β, and that Y_1 and Y_2 are independent. Let $U_1 = Y_1/(Y_1 + Y_2)$ and $U_2 = Y_1 + Y_2$.

a Derive the joint density function for U_1 and U_2.

b Show that the marginal distribution of U_1 is a beta distribution with parameters α_1 and α_2.

c Show that the marginal distribution of U_2 is a gamma distribution with parameters $\alpha = \alpha_1 + \alpha_2$ and β.

d Establish that U_1 and U_2 are independent.

Question 5

Suppose that Y_1 and Y_2 are independent exponentially distributed random variables, both with mean β, and define $U_1 = Y_1 + Y_2$ and $U_2 = Y_1/Y_2$.

a Show that the joint density of (U_1, U_2) is

$$f_{U_1,U_2}(u_1, u_2) = \begin{cases} \dfrac{1}{\beta^2} u_1 e^{-u_1/\beta} \dfrac{1}{(1+u_2)^2}, & 0 < u_1,\ 0 < u_2, \\ 0, & \text{otherwise.} \end{cases}$$

b Are U_1 and U_2 are independent? Why?

Question 6

Let $Y_1, Y_2, \ldots, Y_n$ be independent, uniformly distributed random variables on the interval $[0, \theta]$. Find the

a probability distribution function of $Y_{(n)} = \max(Y_1, Y_2, \ldots, Y_n)$.

b density function of $Y_{(n)}$.

c mean and variance of $Y_{(n)}$.