Logistic regression

Department of Statistics, University of South Carolina

Stat 205: Elementary Statistics for the Biological and Life Sciences

Logistic regression: pp. 538–542

- Sometimes we wish to predict a categorical response Y using a quantitive variable X.
- Consider Y to be binary ($0 =$ failure, $1 =$ success)
- Logistic regression is used to model how the probability of success $\Pr\{Y = 1\}$ depends on X.
- Rather than normally distributed data we now have binomially distributed data.

Esophageal Cancer: Example 12.8.5

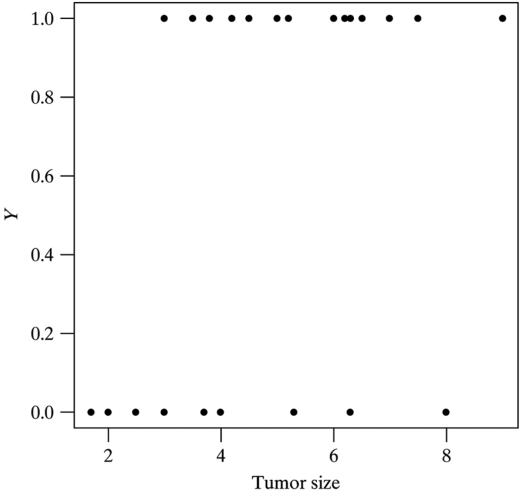
- Esophageal cancer is a serious and very aggressive disease.
- $n = 31$ patients with esophageal cancer studied; looked at size of patient's tumor & whether cancer had spread (metastasized) to lymph nodes.
- Two variables. $Y = 1$ if cancer spread to lymph notes, $Y = 0$ if not. X is maximum dimension (cm) of esophagus tumor.

Esophageal Cancer: Example 12.8.5

Table 12.8.3 Esophageal cancer data

Patient number	Tumor size (cm), X	Lymph node metastasis, Y	Patient number	Tumor size (cm), X	Lymph node metastasis, Y
1	6.5	1	17	6.2	1
2	6.3	0	18	2.0	0
3	3.8	1	19	9.0	1
4	7.5	1	20	4.0	0
5	4.5	1	21	3.0	1
6	3.5	1	22	6.0	1
7	4.0	0	23	4.0	0
8	3.7	0	24	4.0	0
9	6.3	1	25	4.0	0
10	4.2	1	26	5.0	1
11	8.0	0	27	9.0	1
12	5.2	1	28	4.5	1
13	5.0	1	29	3.0	0
14	2.5	0	30	3.0	1
15	7.0	1	31	1.7	0
16	5.3	0			

Plot of Y versus X



Let's group the predictor "Tumor size" into bins (like a histogram) and compute sample proportions for each bin.

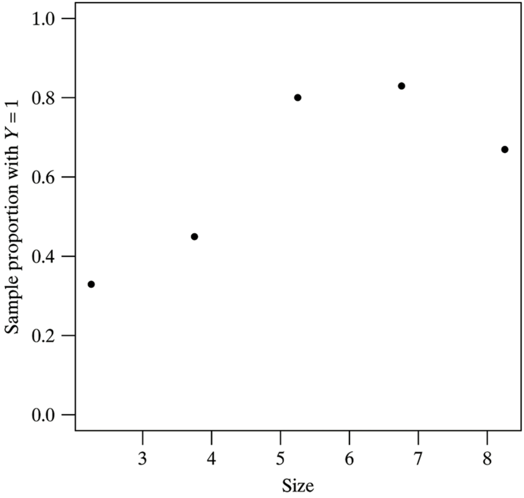
Sample proportions

Table 12.8.4 Esophageal cancer data in groups

Size range	Points with $Y = 1$	Points with $Y = 0$	Fraction $Y = 1$	Proportion $Y = 1$
$(1.5, 3.0]$	2	4	2/6	0.33
$(3.0, 4.5]$	5	6	5/11	0.45
$(4.5, 6.0]$	4	1	4/5	0.80
$(6.0, 7.5]$	5	1	5/6	0.83
$(7.5, 9.0]$	2	1	2/3	0.67

Probability of metastization roughly increases with tumor size.
Let's look at a plot...

Plot of sample proportions



Forms a "lazy S" curve.

Building a model

- The logistic regression model for the probability of success is

$$\Pr\{Y = 1\} = \frac{\exp(\beta_0 + \beta_1 x)}{1 + \exp(\beta_0 + \beta_1 x)}.$$

- R can give us estimates b_0 (for β_0) and b_1 (for β_1), as well as standard errors SE_{b_0} and SE_{b_1} using the function `glm` (instead of `lm` as in regular regression).
- Recall: $\exp(x) = e^x$ where $e \approx 2.718282$, and $\log(x)$ is the natural logarithm; also $\log(e^x) = x$.

R code for cancer data

```
size=c(6.5,6.3,3.8,7.5,4.5,3.5,4.0,3.7,6.3,4.2,8.0,5.2,
       5.0,2.5,7.0,5.3,6.2,2.0,9.0,4.0,3.0,6.0,4.0,4.0,
                4.0,5.0,9.0,4.5,3.0,3.0,1.7)
Y= c(1,0,1,1,1,1,0,0,1,1,0,1,1,0,1,0,1,0,1,0,1,1,
             0,0,0,1,1,1,0,1,0)
> fit=glm(Y~size,family=binomial)
> summary(fit)

Call:
glm(formula = Y ~ size, family = binomial)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.0657  -1.1288   0.5657   0.9844   1.4185

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  -2.0858     1.2256  -1.702   0.0888
size          0.5117     0.2561   1.998   0.0457
```

Building a model

- The estimated probability of whether cancer metastisizes is

$$\Pr\{Y=1\} = \frac{e^{-2.086+0.5117 \text{ size}}}{1+e^{-2.086+0.5117 \text{ size}}}.$$

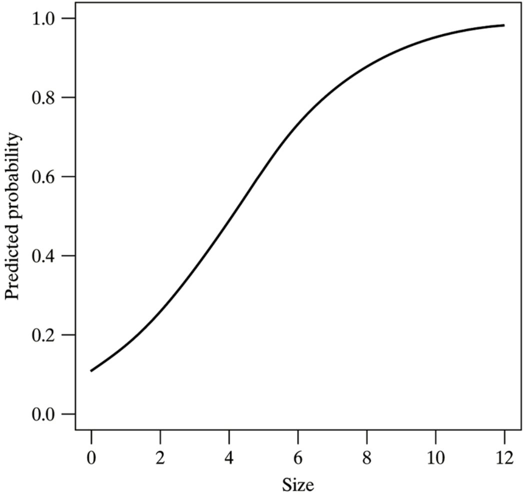
- This is the same as:

$$\log\left(\frac{\Pr\{Y=1\}}{1-\Pr\{Y=1\}}\right) = -2.086 + 0.5117 \text{ size},$$

  the log–odds of metastization.

- Here $b_0 = -2.086$ estimates β_0 and $b_1 = 0.5117$ estimates β_1.
- We test $H_0: \beta_1 = 0$ using the P-value from the table; here P-value$= 0.0457 < 0.05$ so we reject $H_0: \beta_1 = 0$ at the 5% level. There is a significant, positive ($b_1 > 0$) association between tumor size and metastization.

Smooth curve for probability of 'success'

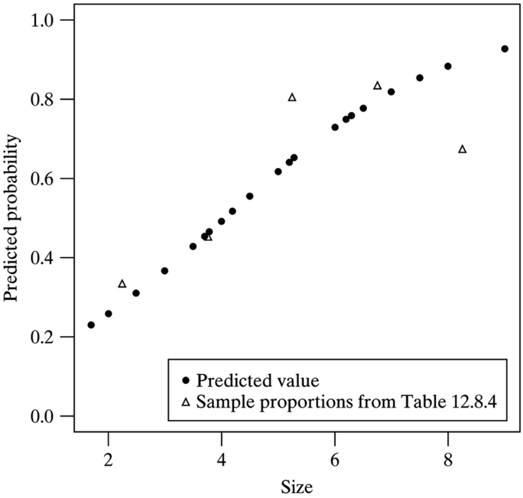


$\Pr\{Y = 1\}$ as a function of tumor size.

Interpretation in terms of odds

- Using the log-odds formula on slide 10, we can show that e^{b_1} is how the odds of success changes when X is increased by one unit.
- For example, when we increase the tumor size by 1 *cm*, the odds of metastization increases by a factor of $e^{0.5117} = 1.668$, i.e. increases by 67%.
- i.e. $e^{0.5117} \approx 1.7$ is an odds ratio.
- If we increase tumor size by 2 *cm* then the odds of metastization increases by $1.7^2 \approx 2.8$ times, or 180%.

Predicted values



Predicted probability at each X-value and sample proportions from windows. Model fits okay.

R code for cancer data (again)

```
size=c(6.5,6.3,3.8,7.5,4.5,3.5,4.0,3.7,6.3,4.2,8.0,5.2,
       5.0,2.5,7.0,5.3,6.2,2.0,9.0,4.0,3.0,6.0,4.0,4.0,
                4.0,5.0,9.0,4.5,3.0,3.0,1.7)
Y= c(1,0,1,1,1,1,0,0,1,1,0,1,1,0,1,0,1,0,1,0,1,1,
             0,0,0,1,1,1,0,1,0)
> fit=glm(Y~size,family=binomial)
> summary(fit)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  -2.0858     1.2256  -1.702   0.0888
size          0.5117     0.2561   1.998   0.0457
```

Questions

- Does the tumor size increase or decrease the odds of having lymph node metastasis Y?
- Is the effect of tumor size significant?
- Find, and interpret a 95% confidence interval for the ratio of odds of Y when the tumor size is increased by 1 cm.

Answers

- The regression coefficient is positive, so increasing the tumor size increases the odds of metastization. This makes intuitive sense. The odds of spreading are increased by a factor of $e^{0.5117} = 1.668$ for every *cm* increase in tumor size.
- The effect is (just) significant, we reject $H_0 : \beta_1$ at the 5% level because $0.0457 < 0.05$.

Confidence interval

- A 95% confidence interval for the log odds ratio is

$$b_1 \pm 1.96 SE_{b_1} = 0.5117 \pm 1.96(0.2561) = (0.010, 1.014).$$

- Exponentiating gives the 95% confidence interval for how the odds change when increasing the size by 1 *cm*: $(e^{0.010}, e^{1.014}) = (1.0097, 2.7557)$.