Stat 704: Homework 3

- **Regression through the origin, again**: Let

$$Y_i = x_i \tau + \epsilon_i,$$

where $E(\epsilon_i) = 0$ and $\text{var}(\epsilon_i) = \sigma^2$.

  (a) Write the model as $\mathbf{Y} = \mathbf{X}\tau + \boldsymbol{\epsilon}$, defining each matrix/vector.

  (b) Show that $\hat{\tau} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} = \frac{\sum_{i=1}^n x_i Y_i}{\sum_{i=1}^n x_i^2}$. A 1×1 matrix is just a number.

  (c) Show $\text{var}(\hat{\tau}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1} = \frac{\sigma^2}{\sum_{i=1}^n x_i^2}$.

- 5.1

- 5.5, 5.13,

- 5.15

- 5.17 **Hint**: If $\text{cov}(\mathbf{Y}) = \mathbf{\Sigma}$ then $\text{cov}(\mathbf{AY}) = \mathbf{A\Sigma A}'$.

- **Extra credit** The normal equations for simple linear regression can be written in matrix terms

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} \sum Y_i \\ \sum x_i Y_i \end{bmatrix}$$

Let

$$\mathbf{X} = \begin{bmatrix} 1 & x_i \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \; \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}.$$

  (a) Verify that the normal equations are equivalent to $[\mathbf{X}'\mathbf{X}]\mathbf{b} = \mathbf{X}'\mathbf{Y}$ and that the solution $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ yields the values of b_0 and b_1 discussed in class and in the book.

  (b) According to results from linear models, the variance of b_0 is the upper left element of $\sigma^2(\mathbf{X}'\mathbf{X})^{-1}$, the variance of b_1 is the lower right, and the covariance between b_0 and b_1 is given by the off-diagonal. Verify that $\sigma^2(\mathbf{X}'\mathbf{X})^{-1}$ gives the correct values (i.e. in the text or notes) for the two variance terms.

  (c) Are b_0 and b_1 independent? Why or why not?