STAT 730: Homework 1

1. Let $\mathbf{X} \in \mathbb{R}^{n \times p}$ where $n > p$ and $\text{rank}(\mathbf{X}) = p$. Let $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ be the orthogonal projection matrix onto $\mathcal{C}(\mathbf{X})$, the column space of $\mathbf{X}$.

   (a) Let $\mathbf{a} \in \mathcal{C}(\mathbf{X})$; show $\mathbf{Pa} = \mathbf{a}$.

   (b) Let $\mathbf{a} \in \mathcal{C}(\mathbf{X})^{\perp}$, the left null space of $\mathbf{X}$. Show that $(\mathcal{I} - \mathbf{P})\mathbf{a} = \mathbf{a}$.

   (c) For any vector $\mathbf{a} \in \mathbb{R}^n$, show $\mathbf{Pa}$ is orthogonal to $(\mathcal{I} - \mathbf{P})\mathbf{a}$. Note that $\mathbf{Pa} + (\mathcal{I} - \mathbf{P})\mathbf{a} = \mathbf{a}$. Furthermore, show directly that $||\mathbf{Pa}||^2 + ||(\mathcal{I} - \mathbf{P})\mathbf{a}||^2 = ||\mathbf{a}||^2$, i.e. the Pythagorean theorem. So $\mathbf{P}$ and $\mathcal{I} - \mathbf{P}$ split $\mathbf{a}$ into two orthogonal pieces, one in $\mathcal{C}(\mathbf{X})$ and one in $\mathcal{C}(\mathbf{X})^{\perp}$.

   (d) Show $\mathbf{P}$ is idempotent.

   (e) Show each column of $\mathbf{X}$ is an unnormalized e-vector of $\mathbf{P}$. What is the corresponding e-value for each? Argue that the p e-vectors can be chosen orthogonal.

   (f) Let $\mathbf{N} \in \mathbb{R}^{n \times (n-p)}$ have orthonormal columns that span $\mathcal{C}(\mathbf{X})^{\perp}$. Argue that the columns of $\mathbf{N}$ are $n - p$ e-vectors; what is the corresponding e-value?

   (g) Use the last two results to write $\mathbf{P} = \mathbf{M}\boldsymbol{\Lambda}\mathbf{M}'$ where $\mathbf{M}$ has e-vectors for columns, $\boldsymbol{\Lambda}$ has e-values down the diagnonal, and $\mathbf{M}'\mathbf{M} = \mathcal{I}_n$. Use this result to show $\text{tr}\mathbf{P} = p$. Hint $\text{tr}\{\mathbf{AB}\} = \text{tr}\{\mathbf{BA}\}$. Argue that $\text{rank}(\mathbf{P}) = p$ as well.

   Note: In general, $\mathbf{P}$ is an othogonal projection matrix $\Leftrightarrow$ it is symmetric and idempotent, i.e. $\mathbf{P}' = \mathbf{P}$ and $\mathbf{PP} = \mathbf{P}$. $\mathbf{P}$ maps any $\mathbf{a}$ onto $\mathcal{C}(\mathbf{P})$.

2. Let $\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$. Use the spectral decomposition theorem to find the unique square root of $\boldsymbol{\Sigma}$, i.e. a symmetric matrix $\boldsymbol{\Sigma}^{1/2}$ s.t. $\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{1/2} = \boldsymbol{\Sigma}$. Also find the inverse of $\boldsymbol{\Sigma}$ using the spectral decomposition. You need to first find the (normalized) eigenvectors and eigenvalues of $\boldsymbol{\Sigma}$.

3. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^p$ be n data vectors. $\mathbf{y}_i = \mathbf{A}\mathbf{x}_i + \mathbf{b}$ where $\mathbf{A} \in \mathbb{R}^{q \times p}$ and $\mathbf{b} \in \mathbb{R}^q$. Show

   $$\bar{\mathbf{y}} = \mathbf{A}\bar{\mathbf{x}} + \mathbf{b}$$

   and

   $$\mathbf{S}_{\mathbf{y}} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{y}_i - \bar{\mathbf{y}})(\mathbf{y}_i - \bar{\mathbf{y}})' = \mathbf{ASA}'.$$

4. Consider two repeated measures versus two independent samples. Let $\mathbf{a} = (-1, 1)'$. Take

$$\mathbf{x}_1, \ldots, \mathbf{x}_n \overset{iid}{\sim} \left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \right).$$

   An estimate of $\delta = \mu_2 - \mu_1 = \mathbf{a}'\boldsymbol{\mu}$ is $d = \mathbf{a}'\bar{\mathbf{x}}$. What is the variance of d? What happens to this variance when $\rho \to 0^+$, which happens when two independent samples are obtained?

   This result shows why repeated measures designs can improve efficiency when looking at differences within individuals. Repeated measures data also requires half as many subjects, although they need to be examined twice.

5. Let $f_1(x_1) = 0.5\lambda^3 x_1^2 e^{-\lambda x_1} I\{x_1 \geq 0\}$ and $f_2(x_2|x_1) = x_1 e^{-x_1 x_2} I\{x_2 \geq 0\}$, i.e. $x_1 \sim \Gamma(3, \lambda)$ and $x_2|x_1 \sim exp(x_1)$. Note $E(x_1) = 3/\lambda$. Find $f(x_1, x_2)$, $\boldsymbol{\mu}$, and $\boldsymbol{\Sigma}$. Feel free to use the fact $E(x_1 x_2) = 1$.

6. On pp. 459–460 the Kronecker product is defined. Prove (IV) on p. 460, $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD})$.

7. MKB 1.5.2.

8. MKB 1.5.3.

9. MKB 2.2.3. Only show that $\frac{-1}{p-1} \leq \alpha \leq 1 \Rightarrow \boldsymbol{\Sigma} \geq 0$. Hint: use Cauchy-Schwartz.