1. Poisson regression. The Ache hunting data set has \(n = 47\) observations \((x_i, d_i, a_i)\) where \(x_i\) is the number of monkeys killed over \(d_i\) days for the \(i\)th hunter, aged \(a_i\) years. It is of interest to estimate and quantify the monkey kill rate as a function of hunter’s age. Hunting prowess confers elevated status among the group, so a natural question is whether hunting ability improves with age, and at which age hunting ability is best. The following R code imports the data, makes the design matrix & log-likelihood function, and fits the model using the \texttt{glm} function to obtain the MLE \(\hat{\theta}\) and covariance matrix. This covariance matrix is based on the expected Fisher information, not observed; they are asymptotically equivalent.

d=read.table(”http://people.stat.sc.edu/hansont/stat740/ache.txt”,header=T)
attach(d) # makes ’age’, ’days’, and ’monkeys’ available to R
n=length(age)
X=cbind(rep(1,n),age,age^2)
ll=function(theta){
  sum(dpois(monkeys,exp(log(days)+X%*%theta),log=T))
}
f=glm(monkeys~age+I(age^2),family=”poisson”,offset=log(days))
f$coef # MLE
vcov(f) # covariance matrix from Fisher scoring

(a) Compute two different sets of crude starting values.
   i. For the “empirical log- rates” \(r_i = \log(\frac{x_i}{d_i}) + 0.1\) use least-squares to obtain \(\theta_0 = (X’X)^{-1}X’r\), and
   ii. A first-order Taylor’s approximation \(e^u \approx 1 + u\) leads to use of least-squares on \(r_i = x_i - 1 - \log d_i\) to obtain \(\theta_0 = (X’X)^{-1}X’r\).
Which starting values are closer to the true MLE?

(b) As shown in class, hand-code Newton-Raphson in R to fit the Poisson regression model
\[
x_{i}^{\text{ind}} \sim \text{Pois}\{\exp\{\log d_i + \theta_1 + \theta_2 a_i + \theta_3 a_i^2\}\}.
\]
Feel free to use \texttt{jacobian} and \texttt{hessian} in the \texttt{numDeriv} R package.

(c) Dr. McMillan was interested in the age at which Ache hunters reached “maximum effectiveness” in hunting, e.g. the age where the mean kill rate reached a maximum. Note that the maximum of \(\exp\{f(u)\}\) occurs at the same \(\hat{u}\) as the maximum of \(f(x)\) due to monotonicity. Find the function \(\tau = g(\theta)\) that is the age where the mean kill-rate (per day) reaches a maximum. Obtain an estimate and 95% CI using the multivariate delta method.
2. For the O-ring failure data and logistic regression model use large-sample normal approximations and the multivariate delta method to get an estimate and 95% CI for the relative risk of failure at 36 degrees (the temperature at launch time) versus 70 degrees, using MLE theory and the Bayesian approach with the normal prior in the R examples. Specifically let the log relative risk be

\[ \tau = g(\theta) = (\theta_1 + 36\theta_2) - \log[1 + \exp(\theta_1 + 36\theta_2)] - (\theta_1 + 70\theta_2) + \log[1 + \exp(\theta_1 + 70\theta_2)]. \]

(a) Use Newton-Raphson as in the example to obtain the MLEs \( \hat{\theta} \) and \( \hat{\tau} = g(\hat{\theta}_1, \hat{\theta}_2) \); use the multivariate delta method to obtain \( se(\hat{\tau}) \). Next, obtain and interpret the MLE of the relative risk \( e^{\hat{\tau}} \) and approximate 95% CI \( (e^{\hat{\tau} - 1.96se(\hat{\tau})}, e^{\hat{\tau} + 1.96se(\hat{\tau})}) \).

(b) Repeat part (a) but instead using the posterior mode and approximate posterior covariance matrix from a Bayesian approach using the bivariate normal prior on \( \theta \) in the R examples. How does the estimate and CI change from the frequentist to the Bayesian approach?

3. Use accept-reject to sample from this bimodal density:

\[ f(x) \propto 3\exp\{-0.5(x+2)^2\} + 7\exp\{-0.5(x-2)^2\}. \]

The normalizing constant is 25.066. For your proposal \( g(\cdot) \), use a \( N(0, 2^2) \) distribution. Verify that your method works via a plot of the true normalized density, the proposal density, and a histogram of the generated values.

4. Use Metropolis-Hastings with an independence \( N(0, 2^2) \) proposal \( g(\cdot) \) to sample from the \( f(\cdot) \) in problem 3. As in problem 3, show that your method works with a plot.