6 Multivariate repeated measures analysis of variance

6.1 Introduction

The statistical model underlying the univariate repeated measures analysis of variance procedures discussed in the last chapter involves a very restrictive assumption about the form of the covariance matrix of a data vector. Specifically, if $\boldsymbol{y}_i$ is the data vector of observations at the n time points from the ith unit, then the model may be written as

$$\boldsymbol{Y}_i' = \boldsymbol{a}_i'\boldsymbol{M} + \boldsymbol{\epsilon}_i', \quad i = 1, \ldots, m, \tag{6.1}$$

where $\boldsymbol{a}_i$ and $\boldsymbol{M}$ are defined in Chapter 5 as, respectively, the $(1 \times q)$ indicator vector of group membership and the $(q \times n)$ matrix whose rows are the transposes of the mean vectors for each group. The error vector e_i associated with the ith unit has, by virtue of the way the model is constructed, covariance matrix

$$\boldsymbol{\Sigma} = \sigma_b^2 \boldsymbol{J}_n + \sigma_e^2 \boldsymbol{I}_n;$$

that is, the model implies the assumption of **compound symmetry**. With the normality assumptions, the model also implies that each data vector has a multivariate normal distribution:

$$\boldsymbol{Y}_i \sim \mathcal{N}_n(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}), \quad \boldsymbol{\mu}_i' = \boldsymbol{a}_i'\boldsymbol{M}.$$

The elements of $\boldsymbol{\mu}_i$ under the model have a very specific form; if unit i is from the ℓth group, the jth element of this vector, $j = 1, \ldots, n$, has the form

$$\mu + \tau_\ell + \gamma_j + (\tau\gamma)_{\ell j}.$$

We saw that, as long as the assumption of compound symmetry is correct, valid tests of statistical hypotheses of interest based on familiar analysis of variance techniques are available. The test of great interest is that of whether there exists a Group by Time interaction, addressing the issue of whether the change in mean response over time differs among groups ("parallelism"). As long as the assumptions of compound symmetry and normality hold, the usual test statistic based on the ratio of two mean squares has an F sampling distribution, so that the value of the statistic may be compared with F critical values to conduct the test. However, if the assumption of compound symmetry does not hold, this is no longer true, and application of the testing procedure may lead to erroneous conclusions.

One approach discussed in Chapter 5 to address this problem was to "adjust" the tests. However, this is a somewhat unsatisfying approach, as it skirts the real problem, which is that the compound symmetry assumption is not appropriate. The simple fact is that this assumption is too restrictive to characterize the kind of correlation patterns that might be seen with longitudinal data. Thus, a more appealing alternative to "adjustment" of tests that are not correct is to return to the statistical model, make a less restrictive assumption, and develop new procedures appropriate for the model under this assumption.

MORE GENERAL MODEL: The most general alternative to the compound symmetry is to go entirely in the opposite direction and assume **very little** about the nature of the covariance structure of a data vector. Recall that in Chapter 5, the deviation $\boldsymbol{\epsilon}_i$ in (6.1) had a very specific form,

$$\boldsymbol{\epsilon}_i' = \mathbf{1}' b_i + \boldsymbol{e}_i',$$

which implied the compound symmetry structure. An alternative view is to consider the model (6.1) as the starting point and make an assumption **directly** about the covariance structure associated with $\boldsymbol{\epsilon}_i$. We may still believe that the covariance matrix of the data vectors $\boldsymbol{Y}_i$ is the same for all i, regardless of group membership; however, we may not believe that this matrix exhibits the compound symmetry structure. We may state this formally by considering the model

$$\boldsymbol{Y}_i' = \boldsymbol{a}_i' \boldsymbol{M} + \boldsymbol{\epsilon}_i', \quad i = 1, \ldots, m, \quad \boldsymbol{\epsilon}_i \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}), \tag{6.2}$$

where $\boldsymbol{\Sigma}$ is now an **arbitrary** covariance matrix assumed to possess **no particular** structure. That is, the most we are willing to say about $\boldsymbol{\Sigma}$ is that it is a symmetric matrix with the **unstructured** form (see Chapter 4)

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix}$$

and is the same for all i.

- This modeling perspective does not explicitly acknowledge how **among-unit** and **within-unit** sources of variation contribute to the overall variation of observations in a data vector. Rather, it is assumed that the aggregate of both sources produces a covariance structure of arbitrary, **unstructured** form; nothing specific about how the two sources combine is characterized.
- The resulting unstructured matrix depends on $n(n+1)/2$ **parameters** (rather than the two parameters σ_b^2 and σ_e^2 under the compound symmetry assumption. Thus, a great many more parameters are required to describe how observations within a data vector vary and covary.

MULTIVARIATE PROCEDURES: With model (6.2) as the starting point, it is possible to develop valid testing procedures for hypotheses of interest. However, the model is much more complicated because there is no longer a nice, simple assumption about covariance. The result is that it is no longer possible as it was under compound symmetry to think on an **individual observation** basis and be able to obtain nice results about ratios of simple mean squares. Thus, familiar procedures based on simple F ratios no longer apply. It is necessary instead to consider the data in the form of **vectors**. Hence, the procedures we now discuss are known as **multivariate** repeated measures analysis of variance methods. This is because they arise as a particular case of a way of thinking about general **multivariate** problems, known as **multivariate analysis of variance** methods (MANOVA). These may be viewed as extensions of usual analysis of variance methods, where now, an "observation" is an entire vector from an unit rather than just a single, scalar response.

PERSPECTIVE: Although a lengthy exposition on multivariate analysis of variance methods and models is possible, we will consider these methods only briefly. A full, general treatment would be found in a full course on multivariate analysis; a typical reference would be Johnson and Wichern (2002).

- This is because, just as the univariate methods of the previous chapter make **too restrictive** an assumption about covariance for many longitudinal data problems, multivariate methods make **too general** an assumption. Indeed, the overall covariance matrix in many longitudinal data settings has some sort of **systematic pattern**.
- The consequence is that they may not be very **powerful** in the statistical sense for detecting departures from null hypotheses of interest, because they must allow for the possibility that the covariance matrix of a data vector may be virtually **anything**! There are now $n(n+1)/2$ parameters defining the covariance structure rather than just 2.
- Thus, the perspective of this instructor is that these methods may be of limited practical utility for longitudinal data problems.

As we will see in subsequent chapters, although we may not be willing to be as narrow as assuming compound symmetry, we may have some basis for assuming **something** about the covariance structure of a data vector, for example, how among- and within-sources of variation affect the response. By taking advantage of what we **are** willing to assume, we may be able to construct more powerful statistical procedures. Moreover, although the model (6.2) gets away from compound symmetry, it still uses a restrictive assumption about the form of the **mean** vector, not incorporating time **explicitly**. Other models we will see later will address all of these issues and lead to more interpretable methods.

6.2 General multivariate problem

GENERAL SET-UP: In order to appreciate the perspective behind the multivariate approach, we consider a general case of a multivariate problem, that usually addressed in a full course on multivariate analysis. Consider the following situation; we use the notation with two subscripts for convenience later.

- Units are randomized into q **groups**.
- Data vector $\boldsymbol{Y}_{h\ell}$ is observed for the hth unit in the ℓth group.
- $\boldsymbol{Y}_{h\ell}$ is assumed to satisfy

$$\boldsymbol{Y}_{h\ell} \sim \mathcal{N}(\boldsymbol{\mu}_\ell, \boldsymbol{\Sigma}),$$

  where $\boldsymbol{\mu}_\ell$ is the mean response vector for group ℓ and $\boldsymbol{\Sigma}$ is an arbitrary covariance matrix assumed to be the **same** for each group.
- There are r_ℓ units in each group, so for group ℓ, $h = 1, \ldots, r_\ell$.
- The components of $\boldsymbol{Y}_{h\ell}$ **may not necessarily** all be measurements of the **same response**. Instead, each component of $\boldsymbol{Y}_{h\ell}$ may represent the measurement of a **different** response. For example, suppose the units are birds of two species. Measurements on n different features of the birds may be taken and collected into a vector $\boldsymbol{Y}_{h\ell}$; e.g. $y_{h\ell 1}$ may be tail length, $y_{h\ell 2}$ may be wing span, $y_{h\ell 3}$ may be body weight, and so on. That is, the elements $Y_{h\ell j}$, $j = 1, \ldots, n$, may consist of measurements of different characteristics.
- Of course, the longitudinal data situation is a special case of this set-up where the $Y_{h\ell j}$ happen to be measurements on the **same** response (over time).

COMPARISON OF INTEREST: Clearly, the main interest is focused on **comparing** the groups on the basis of the responses that make up a data vector somehow.

- Recall in our discussion of univariate methods, we noted that when the responses are all the **same** within a data vector, a natural approach is to think of **averaging** the responses over time and comparing the averages. This was the interpretation of the hypotheses developed for testing the main effect of groups. (Of course, this may be dubious if the profiles are not **parallel**, as discussed in Chapter 5).

- Here, however, it is clear that **averaging** over all responses and comparing the averages across groups would be nonsensical. In the example above, we would be averaging tail length, wing span, body weight, etc, variables that measure entirely different characteristics on different scales!
- Thus, the best we can hope for is to compare all the different responses "simultaneously" somehow. In doing this, it would naturally be important to take into account that observations on the **same unit** are **correlated**.

FORMALLY: In our statistical model, $\boldsymbol{\mu}_\ell$ is the **mean** for data vectors (composed of the n different responses) observed on units in the ℓth group. Thus, we may formally state our desire to compare the n responses "simultaneously" as the desire to compare the q mean vectors $\boldsymbol{\mu}_\ell$, $\ell = 1, \ldots, q$, on the basis of all their components. That is, we are interested in testing the null hypothesis

$$H_0 : \boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_q \tag{6.3}$$

versus the alternative that H_0 is not true. As long as the n responses that make up a data vector are **different** and hence not comparable (e.g. cannot be "averaged"), this is the best we can do to address our general question.

6.3 Hotelling's T^2

The standard methods to test the null hypothesis (6.3) are simply generalizations of standard methods in the case where the data on each unit are just **scalar** observations $y_{h\ell}$, say. That is, $\boldsymbol{Y}_{h\ell}$ is a vector of length $n = 1$. In this section, we give brief statements of these generalizations without much justification. A more in-depth treatment of the general multivariate problem may be found in Johnson and Wichern (1992).

First, consider the case of just $q = 2$ groups.

SCALAR CASE: If the observations were just **scalars** rather than vectors, then we would be interested in the comparison of two **scalar** means μ_ℓ, $\ell = 1, 2$, and H_0 would reduce to

$$H_0 : \mu_1 = \mu_2 \text{ or } \mu_1 - \mu_2 = 0.$$

Furthermore, the unknown covariance matrix $\boldsymbol{\Sigma}$ would reduce to a **single** scalar **variance** value, σ^2, say. Under our normality assumption, the standard test of H_0 would be the two-sample t test.

- Because σ^2 is **unknown**, it must be estimated. This is accomplished by estimating σ^2 based on the observations for each group and then "pooling" the result. That is, letting $\overline{Y}_{\cdot\ell}$ denote the sample mean of the r_ℓ observations $y_{h\ell}$ for group ℓ, find the **sample variance**

$$S_\ell^2 = (r_\ell - 1)^{-1} \sum_{h=1}^{r_\ell} (Y_{h\ell} - \overline{Y}_{\cdot\ell})^2$$

and construct the estimate of σ^2 from data in both groups as the "weighted average"

$$S^2 = (r_1 + r_2 - 2)^{-1}\{(r_1 - 1)S_1^2 + (r_2 - 1)S_2^2\}.$$

- Now, form the test statistic

$$t = \frac{\overline{Y}_{\cdot 1} - \overline{Y}_{\cdot 2}}{\sqrt{(r_1^{-1} + r_2^{-1})s^2}}.$$

The statistic t may be shown to have a Student's t distribution with $r_1 + r_2 - 2$ degrees of freedom.

MULTIVARIATE CASE: The hypothesis is now

$$H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 \text{ or } \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 = \mathbf{0}. \tag{6.4}$$

A natural approach is to seek a multivariate analogue to the t test.

- The analogue of the assumed common variance σ^2 is now the assumed common **covariance matrix $\boldsymbol{\Sigma}$**, which is of course **unknown**. We would like to estimate this matrix for each group and then "pool" the results as in Chapter 4.

- In particular, we may calculate the **pooled sample covariance matrix**. If we collect the sample means $\overline{Y}_{\cdot\ell j}$, $j = 1, \ldots, n$ into a vector

$$\overline{\boldsymbol{Y}}_{\cdot\ell} = \begin{pmatrix} \overline{y}_{\cdot\ell 1} \\ \vdots \\ \overline{y}_{\cdot\ell n} \end{pmatrix},$$

then the sample covariance matrix for group ℓ is the $(n \times n)$ matrix

$$\hat{\boldsymbol{\Sigma}}_\ell = (r_\ell - 1)^{-1} \sum_{h=1}^{r_\ell} (\boldsymbol{Y}_{h\ell} - \overline{\boldsymbol{Y}}_{\cdot\ell})(\boldsymbol{Y}_{h\ell} - \overline{\boldsymbol{Y}}_{\cdot\ell})'. \tag{6.5}$$

Recall that the sum in 6.5) is called a **sum of squares and cross-products** (SS&CP) matrix.

- The overall pooled sample covariance, an estiamtor for $\boldsymbol{\Sigma}$, is then the "weighted average"

$$\hat{\boldsymbol{\Sigma}} = (r_1 + r_2 - 2)^{-1}\{(r_1 - 1)\hat{\boldsymbol{\Sigma}}_1 + (r_2 - 1)\hat{\boldsymbol{\Sigma}}_2\}.$$

- The test statistic analogous to the (square of) the t statistic is known as **Hotelling's** T^2 statistic and is given by

$$T^2 = (r_1^{-1} + r_2^{-1})^{-1}(\overline{\boldsymbol{Y}}_{\cdot 1} - \overline{\boldsymbol{Y}}_{\cdot 2})' \hat{\boldsymbol{\Sigma}}^{-1} (\overline{\boldsymbol{Y}}_{\cdot 1} - \overline{\boldsymbol{Y}}_{\cdot 2}).$$

  It may be shown that

$$\frac{r_1 + r_2 - n - 1}{(r_1 + r_2 - 2)n} T^2 \sim \mathcal{F}_{n, r_1 + r_2 - n - 1}.$$

  Thus, the test of H_0 may be carried out at level α by comparing this version of T^2 to the appropriate α critical value.

  Note that if $n = 1$, the multiplicative factor is equal to 1 and the statistic has an F distribution with 1 and $r_1 + r_2 - 2$ degrees of freedom, which is just the square of the $t_{r_1 + r_2 - 2}$ distribution. That is, the multivariate test reduces to the scalar t test if the dimension of a data vector $n = 1$.

EXAMPLE: For illustration, consider the dental data. Here, the $q = 2$ groups are genders, $r_1 = 11$ (girls), $r_2 = 16$ (boys), and $n = 4$ ages (8, 10, 12, 14). Recall that we found

$$\overline{\boldsymbol{Y}}_{\cdot 1} = (21.182, 22.227, 23.091, 24.091)',$$

$$\overline{\boldsymbol{Y}}_{\cdot 2} = (22.875, 23.813, 25.719, 27.469)'.$$

The estimates of $\boldsymbol{\Sigma}$ for each group are, from Chapter 4,

$$\hat{\boldsymbol{\Sigma}}_1 = \begin{pmatrix} 4.514 & 3.355 & 4.332 & 4.357 \\ 3.355 & 3.618 & 4.027 & 4.077 \\ 4.332 & 4.027 & 5.591 & 5.466 \\ 4.357 & 4.077 & 5.466 & 5.9401 \end{pmatrix},$$

$$\hat{\boldsymbol{\Sigma}}_2 = \begin{pmatrix} 6.017 & 2.292 & 3.629 & 1.613 \\ 2.292 & 4.563 & 2.194 & 2.810 \\ 3.629 & 2.194 & 7.032 & 3.241 \\ 1.613 & 2.810 & 3.241 & 4.349 \end{pmatrix}.$$

The pooled estimate is then easily calculated (Chapter 4) as

$$\hat{\boldsymbol{\Sigma}} = \begin{pmatrix} 5.415 & 2.717 & 3.910 & 2.710 \\ 2.717 & 4.185 & 2.927 & 3.317 \\ 3.910 & 2.927 & 6.456 & 4.131 \\ 2.710 & 3.317 & 4.131 & 4.986 \end{pmatrix}.$$

From these quantities, it is straightforward to calculate

$$\frac{r_1 + r_2 - n - 1}{(r_1 + r_2 - 2)n} T^2 = 3.63,$$

which under our assumptions has an F distribution with 4 and 22 degrees of freedom. $\mathcal{F}_{4,22,0.05} = 2.816$; thus, we would reject H_0 at level $\alpha = 0.05$.

In section 6.6 we will see these calculations done using SAS `PROC GLM`.

HYPOTHESIS IN MATRIX FORM: It is worth noting that the hypothesis in (6.4) may be expressed in the form we have used previously. Specifically, if we define $\boldsymbol{M}$ as before as the $(2 \times n)$ matrix whose rows are the transposed mean vectors $\boldsymbol{\mu}_1'$ and $\boldsymbol{\mu}_2'$, i.e.

$$\boldsymbol{M} = \begin{pmatrix} \mu_{11} & \cdots & \mu_{1n} \\ \mu_{21} & \cdots & \mu_{2n} \end{pmatrix},$$

it should be clear that, defining $\boldsymbol{C} = (1, -1)$, we have (verify)

$$\boldsymbol{CM} = \begin{pmatrix} \mu_{11} - \mu_{21}, & \cdots, & \mu_{1n} - \mu_{2n} \end{pmatrix} = (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)'.$$

Thus, we may express the hypothesis in the form

$$H_0 : \boldsymbol{CMU} = \boldsymbol{0}, \quad \boldsymbol{U} = \boldsymbol{I}_n.$$

6.4 One-way MANOVA

Just as the case of comparing 2 group means for scalar response may be generalized to $q > 2$ groups using analysis of variance techniques, the multivariate analysis above also may be generalized.

SCALAR CASE: Again, if the observations were just **scalars**, we would be interested in the comparison of q **scalar** means μ_ℓ, $\ell = 1, \ldots, q$, and H_0 would reduce to

$$H_0 : \mu_1 = \cdots = \mu_q,$$

and again the unknown covariance matrix $\mathbf{\Sigma}$ would reduce to a **single** scalar **variance** value σ^2. Under the normality assumption, the standard test of H_0 via one-way analysis of variance is based on the **ratio** of two estimators for σ^2. The following is the usual one-way analysis of variance; recall that $m = \sum_{\ell=1}^{q} r_\ell$ is the total number of units:

ANOVA Table

Source	SS	DF	MS	F
Among Groups	$SS_G = \sum_{\ell=1}^{q} r_\ell(\overline{Y}_{\cdot\ell} - \overline{Y}_{\cdot\cdot})^2$	$q-1$	MS_G	MS_G/MS_E
Among-unit Error	$SS_E = \sum_{\ell=1}^{q}\sum_{h=1}^{r_\ell}(Y_{h\ell} - \overline{Y}_{\cdot\ell})^2$	$m-q$	MS_E	
Total	$\sum_{\ell=1}^{q}\sum_{h=1}^{r_\ell}(Y_{h\ell} - \overline{Y}_{\cdot\cdot})^2$	$m-1$		

Note that the "error" sum of squares SS_E may be written as (try it)

$$SS_E = (r_1 - 1)S_1^2 + \cdots + (r_q - 1)S_q^2, \quad S_\ell^2 = (r_\ell - 1)^{-1}\sum_{h=1}^{r_\ell}(Y_{h\ell} - \overline{Y}_{\cdot\ell})^2,$$

where S_ℓ^2 is the sample variance for the ℓth group, so that MS_E has the interpretation as the pooled sample variance estimator for σ^2 across all q groups. MS_G is an estimator for σ^2 based on deviations of the group means from the overall mean, and will overestimate σ^2 if the means are different. It may be shown that the ratio F has sampling distribution that is F with $(q-1)$ and $(m-q)$ degrees of freedom, so that the test is conducted at level α by comparing the calculated value of F to $\mathcal{F}_{q-1,m-q,\alpha}$.

MULTIVARIATE CASE: The hypothesis is now $H_0 : \boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_q$.

As in the case of $q = 2$ groups above, the multivariate generalization involves the fact that there is now an entire covariance matrix $\boldsymbol{\Sigma}$ to estimate rather than just a single variance. Consider the following analogue to the scalar one-way analysis of variance above. Let $\overline{Y}_{\cdot\cdot j}$ be the sample mean of all observations across all units and groups for the jth element and define the **overall** mean vector

$$\overline{\boldsymbol{Y}}_{\cdot\cdot} = \begin{pmatrix} \overline{Y}_{\cdot\cdot 1} \\ \vdots \\ \overline{Y}_{\cdot\cdot n} \end{pmatrix}.$$

MANOVA Table

Source	SS&CP	DF
Among Groups	$\boldsymbol{Q}_H = \sum_{\ell=1}^{q} r_\ell (\overline{\boldsymbol{Y}}_{\cdot\ell} - \overline{\boldsymbol{Y}}_{\cdot\cdot})(\overline{\boldsymbol{Y}}_{\cdot\ell} - \overline{\boldsymbol{Y}}_{\cdot\cdot})'$	$q-1$
Among-unit Error	$\boldsymbol{Q}_E = \sum_{\ell=1}^{q} \sum_{h=1}^{r_\ell} (\boldsymbol{Y}_{h\ell} - \overline{\boldsymbol{Y}}_{\cdot\ell})(\boldsymbol{Y}_{h\ell} - \overline{\boldsymbol{Y}}_{\cdot\ell})'$	$m-q$
Total	$\boldsymbol{Q}_H + \boldsymbol{Q}_E = \sum_{\ell=1}^{q} \sum_{h=1}^{r_\ell} (\boldsymbol{Y}_{h\ell} - \overline{\boldsymbol{Y}}_{\cdot\cdot})(\boldsymbol{Y}_{h\ell} - \overline{\boldsymbol{Y}}_{\cdot\cdot})'$	$m-1$

Comparing the entries in this table to those in the scalar ANOVA table, we see that they appear to be multivariate generalizations. In particular, the entries are now **matrices**. Each may be viewed as an attempt to estimate $\boldsymbol{\Sigma}$.

It is straightforward to verify (try it) that the Among-unit Error sum of squares and cross products matrix $\boldsymbol{Q}_E$ may be written

$$\boldsymbol{Q}_E = (r_1 - 1)\hat{\boldsymbol{\Sigma}}_1 + \cdots + (r_q - 1)\hat{\boldsymbol{\Sigma}}_q,$$

where $\hat{\boldsymbol{\Sigma}}_\ell$ is the estimate (6.5) of $\boldsymbol{\Sigma}$ based on the data vectors from group ℓ. Thus, just as in the scalar case, this quantity divided by its degrees of freedom has the interpretation as a "pooled" estimate of $\boldsymbol{\Sigma}$ across groups.

MULTIVARIATE TESTS: Unfortunately, because these entries are matrices, it is no longer straightforward to construct a unique generalization of the F ratio that may be used to test H_0. Clearly, one would like to compare the "magnitude" of the SS&CP matrices $\boldsymbol{Q}_H$ and $\boldsymbol{Q}_E$ somehow, but there is no one way to do this. There are a number of statistics that have been proposed based on these quantities that have this interpretation.

- The most commonly discussed statistic is known as **Wilks' lambda** and may be motivated informally as follows. In the scalar case, the F ratio is

$$\frac{SS_G/(q-1)}{SS_E/(m-q)};$$

thus, in the scalar case, H_0 is rejected when the ratio SS_G/SS_E is large. This is equivalent to rejecting for large values of $1 + SS_G/SS_E$ or small values of

$$\frac{1}{1+SS_G/SS_E} = \frac{SS_E}{SS_G+SS_E}.$$

For the multivariate problem, the Wilks' lambda statistic is the analogue of this quantity,

$$T_W = \frac{|\boldsymbol{Q}_E|}{|\boldsymbol{Q}_H+\boldsymbol{Q}_E|};$$

here, the **determinant** of each SS&CP matrix is taken, reducing the matrix to a single number. This number is often referred to as the **generalized sample variance**; see Johnson and Wichern (2002) for a deeper discussion. One rejects H_0 for small values of T_W (how small will be discussed in a moment).

- Another statistic is referred to as the **Lawley-Hotelling trace**; reject H_0 for large values of

$$T_{LH} = \text{tr}(\boldsymbol{Q}_H\boldsymbol{Q}_E^{-1}).$$

- Other statistics are **Pillai's trace** and **Roy's greatest root**.

- None of these approaches been shown to be superior to the others in general. In addition, all are equivalent to using the Hotelling T^2 statistic in the case $q = 2$.

A full discussion of the theoretical underpinnings of these methods is beyond the scope of our discussion. Here, we note briefly the salient points:

- It is possible in certain special cases to work out the exact sampling distribution of these statistics. As mentioned above, when $q = 2$ and we are testing whether the two means are the same, all of these statistics may be shown to be the same and equivalent to conducting the test based on Hotelling's T^2 statistics.

- When $n = 1, 2$ and $q \geq 2$ or when $n \geq 1$ and $q = 2, 3$, it is possible to show that certain functions of T_W have an F sampling distribution, and this may be used to conduct the test **exactly**. These are listed in Johnson and Wichern (2002).

- In other situations, it is possible to show that the sampling distributions may be **approximated** by F or other distributions.

- SAS `PROC GLM` calculates all of these statistics and provides either exact or approximate p-values, depending on the situation.

We will consider the application of these methods to the dental study data and the guinea pig diet data in section 6.6.

HYPOTHESIS IN MATRIX FORM: It is again worth noting that the hypothesis of interest (6.3) may be expressed in the form $H_0 : \boldsymbol{CMU} = \mathbf{0}$ for suitable choice of $\boldsymbol{C}$ and with $\boldsymbol{U} = \boldsymbol{I}_n$. For example, consider the case $q = 3$, with

$$\boldsymbol{M} = \begin{pmatrix} \mu_{11} & \cdots & \mu_{1n} \\ \mu_{21} & \cdots & \mu_{2n} \\ \mu_{31} & \cdots & \mu_{3n} \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \tag{6.6}$$

$$\boldsymbol{CM} = \begin{pmatrix} \mu_{11} - \mu_{21} & \cdots & \mu_{1n} - \mu_{2n} \\ \mu_{11} - \mu_{31} & \cdots & \mu_{1n} - \mu_{3n} \end{pmatrix} = \begin{pmatrix} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)' \\ (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_3)' \end{pmatrix}.$$

Setting this equal to $\mathbf{0}$ may thus be seen to be equivalent to saying that all of the mean vectors $\boldsymbol{\mu}_\ell$ are the same.

SUMMARY: We have seen that, in situations where a data vector consists of n observations on possibly **different** characteristics on **different scales**, it is possible to test whether the entire **mean vectors** for each group are the same using what are usually called one-way MANOVA methods.

- If the null hypothesis (6.3) is rejected, then this means we have evidence to suggest that at least one of the q mean vectors differs from the others in at least one of the n components. This is not particularly informative, particularly if q and/or n are somewhat large.

- In addition, it seems intuitively that it would be difficult to detect such a difference – with q vectors and n components, there are a lot of comparisons that must be taken into account when looking for a difference.

- Furthermore, the methods are requiring estimation of all $n(n+1)/2$ elements of the (assumed common across groups) covariance matrix $\boldsymbol{\Sigma}$.

- Thus, the basis for our earlier remark that multivariate procedures may lack power for detecting differences should now be clear.

- Furthermore, when the n elements of a data vector are all observations on the **same** characteristic as in the case of longitudinal data, these methods do not seem to really get at the heart of matters. Focusing on H_0 in (6.3) ignores the questions of interest, such as that of **parallelism**.

6.5 Profile Analysis

It turns out that one can conduct more focused multivariate tests that make no particular assumption about the form of $\boldsymbol{\Sigma}$. Recall that the MANOVA test of (6.3), $H_0 : \boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_q$ could be regarded as testing a particular hypothesis of the form

$$H_0 : \boldsymbol{CMU} = \boldsymbol{0}$$

for suitable choice of $\boldsymbol{C}$ and with $\boldsymbol{U} = \boldsymbol{I}_n$. It should thus come as no surprise that it is possible to develop such multivariate procedures for more general choices of $\boldsymbol{C}$ and $\boldsymbol{U}$.

HYPOTHESIS OF PARALLELISM: Of particular interest in the case of longitudinal data is the test of **parallelism** or **group by time interaction**. In the last chapter, we saw that the null hypothesis corresponding to parallelism could be expressed in terms of the elements of the mean vectors $\boldsymbol{\mu}_\ell$ or equivalently in terms of the $taugam_{\ell j}$:

$$H_0 : \text{ all } (\tau\gamma)_{\ell j} = 0.$$

In particular, in the case of $q = 2$ and $n = 3$, we saw that this test could be represented with

$$\boldsymbol{C} = \begin{pmatrix} 1 & -1 \end{pmatrix}, \quad \boldsymbol{U} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \end{pmatrix}.$$

For general q and n, we may write this in a streamlined fashion. If we let $\boldsymbol{j}_p$ denote a column vector of 1's of length p, then (try it!) choosing

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{j}_{q-1} & -\boldsymbol{I}_{q-1} \end{pmatrix} \; (q-1 \times q), \quad \boldsymbol{U} = \begin{pmatrix} \boldsymbol{j}'_{n-1} \\ -\boldsymbol{I}_{n-1} \end{pmatrix} \; (n \times n-1) \tag{6.7}$$

gives the null hypothesis of parallelism.

MULTIVARIATE TEST FOR PARALLELISM: Recall that the **univariate** test of this null hypothesis discussed in Chapter 5 was predicated on the assumption of **compound symmetry**. Here, we seek a test in the same spirit of those in the last section that make no assumption about the form of $\boldsymbol{\Sigma}$.

To understand this, we first consider the multivariate test of (6.3). Recall in the MANOVA table of the last section that this test boiled down to making a comparison between 2 SS&CP matrices, $\boldsymbol{Q}_H$ and $\boldsymbol{Q}_E$ that focused on the particular issue of the hypothesis.

- $\boldsymbol{Q}_E$ effectively measured the distance of individual data vectors from the means for their group.
- $\boldsymbol{Q}_H$ measured the distance of group mean vectors from the overall mean vector.
- We would expect $\boldsymbol{Q}_H$ to be "large" relative to $\boldsymbol{Q}_E$ if there really were a difference among the q means $\boldsymbol{\mu}_\ell$, $\ell = 1 \ldots, q$.

We would clearly like to do something **similar** for the null hypothesis of parallelism.

HEURISTIC DESCRIPTION: It turns out that for the test of (6.3), $H_0 : \boldsymbol{\mu}_1 = \ldots = \boldsymbol{\mu}_q$, which may be expressed in the form $H_0 : \boldsymbol{CMU} = \boldsymbol{0}$ with $\boldsymbol{C}$ as in (6.6) and $\boldsymbol{U} = \boldsymbol{I}_n$, we may express $\boldsymbol{Q}_H$ and $\boldsymbol{Q}_E$ in an alternative form as functions of $\boldsymbol{C}$, $\boldsymbol{M}$, and $\boldsymbol{U}$ ($= \boldsymbol{I}_n$ here). Specifically, recall that we may express the underlying statistical model as in (6.1), i.e.

$$\boldsymbol{Y}_i' = \boldsymbol{a}_i'\boldsymbol{M} + \boldsymbol{\epsilon}_i', \quad i = 1, \ldots, m.$$

We saw in Chapter 5 that this may be written more succinctly as (5.14), i.e.

$$\boldsymbol{\mathcal{Y}} = \boldsymbol{AM} + \boldsymbol{\epsilon},$$

where $\boldsymbol{\mathcal{Y}}$ is the $(m \times n)$ matrix with rows $\boldsymbol{Y}_i'$ and similarly for $\boldsymbol{\epsilon}$, and $\boldsymbol{A}$ $(m \times q)$ has rows $\boldsymbol{a}_i'$. It is an exercise in matrix algebra to show that we may write $\boldsymbol{Q}_H$ and $\boldsymbol{Q}_E$ in terms of this model as

$$\boldsymbol{Q}_H = (\boldsymbol{C}\widehat{\boldsymbol{M}}\boldsymbol{U})'\{\boldsymbol{C}(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{C}'\}^{-1}(\boldsymbol{C}\widehat{\boldsymbol{M}}\boldsymbol{U}) \tag{6.8}$$

$$\boldsymbol{Q}_E = \boldsymbol{U}'\boldsymbol{\mathcal{Y}}'\{\boldsymbol{I}_n - \boldsymbol{A}(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\}\boldsymbol{\mathcal{Y}}\boldsymbol{U} \tag{6.9}$$

with

$$\widehat{\boldsymbol{M}} = (\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\mathcal{Y}}, \quad \boldsymbol{U} = \boldsymbol{I}_n.$$

A technical justification of (6.8) and (6.9) may be found in, for example, Vonesh and Chinchilli (1997, p. 50); they show that this representation and the form of the Wilks' lambda statistic T_W may be derived using the principles of **maximum likelihood**, which we will discuss later in the course in a different context.

The above results are in fact valid for **any** suitable choice of $\boldsymbol{C}$ and $\boldsymbol{U}$, such as those corresponding to the null hypothesis of parallelism.

- That is, for a null hypothesis of the form $H_0 : \boldsymbol{CMU} = \boldsymbol{0}$, one may construct corresponding SS&CP matrices $\boldsymbol{Q}_H$ and $\boldsymbol{Q}_E$. These are often called the **hypothesis** and **error** SS&CP matrices, respectively.
- One may then construct any of the test statistics such as Wilks' lambda T_W discussed in the last section. It may be shown that these will provide either approximate or exact tests, depending on the circumstances, for the null hypothesis corresponding to the choice of $\boldsymbol{C}$ and $\boldsymbol{U}$.
- These test are **multivariate** in the sense that **no assumption** of a particular structure for $\boldsymbol{\Sigma}$ is made.

PROFILE ANALYSIS: In the particular context of repeated measurement data, where the n observations in a data vector are all on the same characteristic, conducting appropriate **multivariate** tests for parallelism and other issues of interest is known as **profile analysis.** This is usually carried out in practice as follows.

- The test of primary interest is that of **parallelism** or Group by Time interaction. This may be represented in the form $H_0 : \boldsymbol{CMU} = \boldsymbol{0}$ with $\boldsymbol{C}$ and $\boldsymbol{U}$ as in(6.7), so that suitable $\boldsymbol{Q}_H$ and $\boldsymbol{Q}_E$ may be calculated. Thus, test statistics such as Wilks' lambda, Pillai's trace, and so on may be used to conduct the test. Depending on the dimensions q and n, these tests may be exact or approximate and may or may not coincide.
- The next test is usually only conducted if the hypothesis of parallelism is not rejected.

  The test of $H_0 : \boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_q$ may be written in the form $H_0 : \boldsymbol{CMU} = \boldsymbol{0}$ with $\boldsymbol{C}$ as in (6.7) $\boldsymbol{U} = \boldsymbol{I}_n$. This is just the usual MANOVA test discussed in the last section; when repeated measurements are involved, this test is often called the test for **coincidence**. Clearly, if the profiles are **not parallel**, then testing coincidence seems ill-advised, as it is not clear what it means.

As we discussed in Chapter 5, if the profiles **are parallel**, then it turns out that we may refine this test. Specifically, it may be shown that testing this H_0 with the **additional** assumption that the profiles are **parallel** is equivalent to testing the hypothesis $H_0 : \boldsymbol{CMU} = \boldsymbol{0}$ with $\boldsymbol{C}$ as in (6.7) but with $\boldsymbol{U} = \boldsymbol{j}_n/n$. Note that this is exactly the same hypothesis we discussed in Chapter 5 – if the profiles are parallel, then testing whether they in fact coincide is the same as testing whether the **averages** of the means over time is the same for each group; that is, the test we called **main effect of group**.

It turns out that, for testing this hypothesis, the **multivariate** tests are all equivalent. Furthermore, they reduce to the **univariate** F test for the **main effect of groups** we discussed in Chapter 5! Intuitively, this makes sense – we are basing the test on **averaging** observations over time, thus effectively "distilling" the data for each unit down to a single average. The "distilling" operation averages across **time**, so how observations within a data vector are **correlated** is being "averaged away." As long as $\boldsymbol{\Sigma}$ is the same for all data vectors, these "distilled" data are all have the same variance, so we would expect an ordinary F ratio to apply.

- This test is also usually conducted only if the hypothesis of parallelism is not rejected.

  It is also of interest to know whether the profiles are in fact **constant** over time. It may be shown (try it!) that this may be represented in the form $H_0 : \boldsymbol{CMU} = \boldsymbol{0}$ with $\boldsymbol{U}$ as in (6.7) and $\boldsymbol{C} = \boldsymbol{I}_q$. As with the test for coincidence, if the profiles are **not parallel**, then testing whether they are **constant** over time seems inappropriate.

  If there is strong evidence of **parallelism**, then we may refine this test also. It may be shown that testing H_0 for **constancy** with the **additional** assumption that the profiles are **parallel** is equivalent to testing $H_0 : \boldsymbol{CMU} = \boldsymbol{0}$ with the choices $\boldsymbol{U}$ as in (6.7) and $\boldsymbol{C} = \boldsymbol{j}_q'/q$, a $(1 \times q)$ vector of $1/q$'s. Note (try it) that this is the exactly the same hypothesis discussed for the **main effect of time** discussed in Chapter 5 – if we know the profiles are parallel, then asking whether the means are constant over time is the same as asking whether the mean response **averaged across groups** is the same at each time.

  It turns out that, for testing this hypothesis, the **multivariate** tests are again all equivalent. **However**, the multivariate test is **different** from the **univariate** tests. Intuitively, this also makes sense – we are basing the test on **averaging** observations across **groups**. Thus, although we are again "distilling" the data, we are now doing it over groups, so that **time**, and how observations are **correlated** over time, is not being "averaged away." As a result, what is being assumed about the form of $\boldsymbol{\Sigma}$ still plays a role.

The (common) multivariate test statistic boils down to a statistic that is a generalization of the form of the Hotelling's T^2 statistic, and it may be shown that this statistic multiplied by a suitable factor thus has exactly an F distribution. It is important to recognize that, although both the **univariate** and **multivariate** test statistics both have F sampling distributions, they are **different** tests, being based on different assumptions on the form of $\mathbf{\Sigma}$. Which one is more appropriate depends on the true form of $\mathbf{\Sigma}$.

6.6 Implementation with SAS

We consider again the two examples of Chapter 5:

1. The dental study data. Here, $q = 2$ and $n = 4$, with the "time" factor being the age of the children and equally-spaced "time" points at 8, 10, 12, and 14 years of age.

2. the guinea pig diet data. Here, $q = 3$ and $n = 6$, with the "time" factor being weeks and unequally-spaced "time" points at 1, 3, 4, 5, 6, and 7 weeks.

In each case, we use SAS `PROC GLM` and its various options to carry out both the one-way MANOVA analysis comparing the group mean vectors and the refined hypotheses of **profile analysis**. These examples thus serve to illustrate how this SAS procedure may be used to conduct multivariate repeated measures analysis of variance.

EXAMPLE 1 – DENTAL STUDY DATA: The data are read in from the file `dental.dat`.
PROGRAM:

```
/*****************************************************************
  CHAPTER 6, EXAMPLE 1

  Analysis of the dental study data by multivariate repeated
  measures analysis of variance using PROC GLM

  -  the repeated measurement factor is age (time)

  -  there is one "treatment" factor, gender

*****************************************************************/

options ls=80 ps=59 nodate; run;

/*****************************************************************

  See Example 1 in Chapter 4 for the form of the input data set.
  It is not in the correct from for the analysis; thus we create
  a new data set such that each record in the data set represents
  the observations from a different unit.

*****************************************************************/

data dent1; infile 'dental.dat';
  input obsno child age distance gender;
run;
```

```
data dent1; set dent1;
  if age=8 then age=1;
  if age=10 then age=2;
  if age=12 then age=3;
  if age=14 then age=4;
  drop obsno;
run;

proc sort data=dent1;
  by gender child;
data dent2(keep=age1-age4 gender);
  array aa{4} age1-age4;
  do age=1 to 4;
  set dent1;
  by gender child;
  aa{age}=distance;
  if last.child then return;
end;
run;

/*****************************************************************

  The sample mean vectors for each gender were found in Example 1
  of Chapter 4.  Here, we use PROC CORR to calculate the estimates
  of Sigma, the assumed common covariance matrix, separately for
  each group.  The COV option asks for the covariance matrix
  to be printed.

*****************************************************************/

proc sort data=dent2; by gender; run;
proc corr data=dent2 cov; by gender; var age1 age2 age3 age4; run;

/*****************************************************************

  Use PROC GLM to carry out the multivariate analysis.

  First, call PROC GLM and use the MANOVA statement to get the
  MANOVA test of equality of gender means.  Here, this is
  equivalent to Hotelling's T^2 test because there are 2 groups.

  The PRINTH and PRINTE options print the SS&CP matrices
  Q_H and Q_E corresponding to the null hypothesis of equal means.

  The option NOUNI suppresses individual analyses of variance
  for the data at each age value from being printed.  Without
  the NOUNI option in the MODEL statement, note that PROC GLM does
  a separate univariate ANOVA on the data at each age separately.

*****************************************************************/

proc glm data=dent2;
  class gender;
  model age1 age2 age3 age4 = gender;
  manova h=gender / printh printe;

/*****************************************************************

  Now use the REPEATED option to do profile analysis.  The
  "between subjects" (units) test is that for coincidence assuming
  profiles are parallel, based on averaging across times.
  Thus, as discussed in section 5.5, it is the same as the
  univariate test.

  The tests for age and age*gender resulting from this analysis
  are the multivariate tests for profile constancy and parallelism,
  respectively.  The test for constancy (age effect here) is the
  multivariate test for constancy assuming that the profiles are
  parallel, as discussed in section 5.5  Both of these tests are
  different from the corresponding univariate tests we saw in
  section 4.8 that are based on the assumption of compound symmetry.

  The NOU option in the REPEATED statement suppresses printing of the
  univariate tests of these factors.

  The within-unit analyses using different contrast matrices will
  be the same as in the univariate case (see the discussion in
  section 4.6.  Thus, we do not do this analysis here.

*****************************************************************/

proc glm data=dent2;
  class gender;
  model age1 age2 age3 age4 = gender / nouni;
  repeated age / nou;
```

OUTPUT:

```
                                                                    1

---------------------------------- gender=0 ----------------------------------

                          The CORR Procedure

            4  Variables:    age1     age2     age3     age4

                       Covariance Matrix, DF = 10

                  age1             age2             age3             age4

 age1       4.513636364      3.354545455      4.331818182      4.356818182
 age2       3.354545455      3.618181818      4.027272727      4.077272727
 age3       4.331818182      4.027272727      5.590909091      5.465909091
 age4       4.356818182      4.077272727      5.465909091      5.940909091

                          Simple Statistics

Variable          N        Mean     Std Dev          Sum     Minimum     Maximum

age1             11    21.18182     2.12453   233.00000    16.50000    24.50000
age2             11    22.22727     1.90215   244.50000    19.00000    25.00000
age3             11    23.09091     2.36451   254.00000    19.00000    28.00000
age4             11    24.09091     2.43740   265.00000    19.50000    28.00000

                  Pearson Correlation Coefficients, N = 11
                        Prob > |r| under H0: Rho=0

                       age1          age2          age3          age4

         age1       1.00000       0.83009       0.86231       0.84136
                                     0.0016        0.0006        0.0012

         age2       0.83009       1.00000       0.89542       0.87942
                     0.0016                      0.0002        0.0004

         age3       0.86231       0.89542       1.00000       0.94841
                     0.0006        0.0002                      <.0001

         age4       0.84136       0.87942       0.94841       1.00000
                     0.0012        0.0004        <.0001

                                                                    2

---------------------------------- gender=1 ----------------------------------

                          The CORR Procedure

            4  Variables:    age1     age2     age3     age4

                       Covariance Matrix, DF = 15

                  age1             age2             age3             age4

 age1       6.016666667      2.291666667      3.629166667      1.612500000
 age2       2.291666667      4.562500000      2.193750000      2.810416667
 age3       3.629166667      2.193750000      7.032291667      3.240625000
 age4       1.612500000      2.810416667      3.240625000      4.348958333

                          Simple Statistics

Variable          N        Mean     Std Dev          Sum     Minimum     Maximum

age1             16    22.87500     2.45289   366.00000    17.00000    27.50000
age2             16    23.81250     2.13600   381.00000    20.50000    28.00000
age3             16    25.71875     2.65185   411.50000    22.50000    31.00000
age4             16    27.46875     2.08542   439.50000    25.00000    31.50000

                  Pearson Correlation Coefficients, N = 16
                        Prob > |r| under H0: Rho=0

                       age1          age2          age3          age4

         age1       1.00000       0.43739       0.55793       0.31523
                                     0.0902        0.0247        0.2343

         age2       0.43739       1.00000       0.38729       0.63092
                     0.0902                      0.1383        0.0088

         age3       0.55793       0.38729       1.00000       0.58599
```

```
                  0.0247        0.1383                    0.0171

        age4      0.31523       0.63092       0.58599     1.00000
                  0.2343        0.0088        0.0171
```

3

The GLM Procedure

Class Level Information

Class	Levels	Values
gender	2	0 1

Number of observations 27

4

The GLM Procedure

Dependent Variable: age1

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	1	18.6877104	18.6877104	3.45	0.0750
Error	25	135.3863636	5.4154545		
Corrected Total	26	154.0740741			

R-Square	Coeff Var	Root MSE	age1 Mean
0.121290	10.48949	2.327113	22.18519

Source	DF	Type I SS	Mean Square	F Value	Pr > F
gender	1	18.68771044	18.68771044	3.45	0.0750

Source	DF	Type III SS	Mean Square	F Value	Pr > F
gender	1	18.68771044	18.68771044	3.45	0.0750

5

The GLM Procedure

Dependent Variable: age2

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	1	16.3806818	16.3806818	3.91	0.0590
Error	25	104.6193182	4.1847727		
Corrected Total	26	121.0000000			

R-Square	Coeff Var	Root MSE	age2 Mean
0.135378	8.830238	2.045672	23.16667

Source	DF	Type I SS	Mean Square	F Value	Pr > F
gender	1	16.38068182	16.38068182	3.91	0.0590

Source	DF	Type III SS	Mean Square	F Value	Pr > F
gender	1	16.38068182	16.38068182	3.91	0.0590

6

The GLM Procedure

Dependent Variable: age3

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	1	45.0139415	45.0139415	6.97	0.0141
Error	25	161.3934659	6.4557386		

```
Corrected Total            26    206.4074074

              R-Square     Coeff Var      Root MSE     age3 Mean

              0.218083      10.30834      2.540815      24.64815

Source                     DF      Type I SS    Mean Square   F Value   Pr > F

gender                      1    45.01394150    45.01394150      6.97   0.0141

Source                     DF    Type III SS    Mean Square   F Value   Pr > F

gender                      1    45.01394150    45.01394150      6.97   0.0141
                                                                      7

                              The GLM Procedure

Dependent Variable: age4

                                          Sum of
Source                     DF        Squares    Mean Square   F Value   Pr > F

Model                       1     74.3750526     74.3750526     14.92   0.0007

Error                      25    124.6434659      4.9857386

Corrected Total            26    199.0185185

              R-Square     Coeff Var      Root MSE     age4 Mean

              0.373709      8.557512      2.232877      26.09259

Source                     DF      Type I SS    Mean Square   F Value   Pr > F

gender                      1    74.37505261    74.37505261     14.92   0.0007

Source                     DF    Type III SS    Mean Square   F Value   Pr > F

gender                      1    74.37505261    74.37505261     14.92   0.0007
                                                                      8

                              The GLM Procedure
                        Multivariate Analysis of Variance

                              E = Error SSCP Matrix

                   age1              age2              age3              age4

age1      135.38636364      67.920454545      97.755681818      67.755681818
age2      67.920454545      104.61931818      73.178977273      82.928977273
age3      97.755681818      73.178977273      161.39346591      103.26846591
age4      67.755681818      82.928977273      103.26846591      124.64346591

   Partial Correlation Coefficients from the Error SSCP Matrix / Prob > |r|

     DF = 25           age1              age2              age3              age4

     age1          1.000000          0.570699          0.661320          0.521583
                                       0.0023            0.0002            0.0063

     age2          0.570699          1.000000          0.563167          0.726216
                     0.0023                              0.0027            <.0001

     age3          0.661320          0.563167          1.000000          0.728098
                     0.0002            0.0027                              <.0001

     age4          0.521583          0.726216          0.728098          1.000000
                     0.0063            <.0001            <.0001
                                                                      9

                              The GLM Procedure
                        Multivariate Analysis of Variance

                         H = Type III SSCP Matrix for gender

                   age1              age2              age3              age4

age1      18.687710438      17.496212121      29.003577441      37.281355219
age2      17.496212121      16.380681818      27.154356061      34.904356061
```

age3	29.003577441	27.154356061	45.013941498	57.861163721
age4	37.281355219	34.904356061	57.861163721	74.375052609

Characteristic Roots and Vectors of: E Inverse * H, where
H = Type III SSCP Matrix for gender
E = Error SSCP Matrix

Characteristic Root	Percent	Characteristic Vector V'EV=1 age1	age2	age3	age4
0.66030051	100.00	0.01032388	-0.04593889	-0.01003125	0.11841126
0.00000000	0.00	-0.07039943	0.13377597	0.00249339	-0.02943257
0.00000000	0.00	-0.08397385	-0.01167207	0.12114416	-0.04667529
0.00000000	0.00	0.05246789	0.05239507	0.05062221	-0.09027154

MANOVA Test Criteria and Exact F Statistics for
the Hypothesis of No Overall gender Effect
H = Type III SSCP Matrix for gender
E = Error SSCP Matrix

S=1 M=1 N=10

Statistic	Value	F Value	Num DF	Den DF	Pr > F
Wilks' Lambda	0.60230061	3.63	4	22	0.0203
Pillai's Trace	0.39769939	3.63	4	22	0.0203
Hotelling-Lawley Trace	0.66030051	3.63	4	22	0.0203
Roy's Greatest Root	0.66030051	3.63	4	22	0.0203

10

The GLM Procedure

Class Level Information

Class	Levels	Values
gender	2	0 1

Number of observations 27

11

The GLM Procedure
Repeated Measures Analysis of Variance

Repeated Measures Level Information

Dependent Variable	age1	age2	age3	age4
Level of age	1	2	3	4

Manova Test Criteria and Exact F Statistics for the Hypothesis of no age Effect
H = Type III SSCP Matrix for age
E = Error SSCP Matrix

S=1 M=0.5 N=10.5

Statistic	Value	F Value	Num DF	Den DF	Pr > F
Wilks' Lambda	0.19479424	31.69	3	23	<.0001
Pillai's Trace	0.80520576	31.69	3	23	<.0001
Hotelling-Lawley Trace	4.13362211	31.69	3	23	<.0001
Roy's Greatest Root	4.13362211	31.69	3	23	<.0001

Manova Test Criteria and Exact F Statistics
for the Hypothesis of no age*gender Effect
H = Type III SSCP Matrix for age*gender
E = Error SSCP Matrix

S=1 M=0.5 N=10.5

Statistic	Value	F Value	Num DF	Den DF	Pr > F
Wilks' Lambda	0.73988739	2.70	3	23	0.0696
Pillai's Trace	0.26011261	2.70	3	23	0.0696
Hotelling-Lawley Trace	0.35155702	2.70	3	23	0.0696
Roy's Greatest Root	0.35155702	2.70	3	23	0.0696

12

The GLM Procedure
Repeated Measures Analysis of Variance
Tests of Hypotheses for Between Subjects Effects

```
 Source                     DF    Type III SS    Mean Square   F Value   Pr > F

 gender                      1    140.4648569    140.4648569      9.29   0.0054
 Error                      25    377.9147727     15.1165909
```

EXAMPLE 2 – GUINEA PIG DIET DATA: The data are read in from the file `diet.dat`.

PROGRAM:

```
/*****************************************************************

  CHAPTER 6, EXAMPLE 2

  Analysis of the vitamin E data by multivariate repeated
  measures analysis of variance using PROC GLM

  -  the repeated measurement factor is week (time)

  -  there is one "treatment" factor, dose

*****************************************************************/

options ls=80 ps=59 nodate; run;

/*****************************************************************

  The data set is shown in Example 2 of Chapter 5.  It is
  already in the form required for PROC GLM to perform the
  multivariate analysis; that is, each line in the data set
  contains all the data for a given unit.  Thus,
  we need only input the data as is and do not need to create
  a new data set.

*****************************************************************/

data pigs1; infile 'diet.dat';
  input pig week1 week3 week4 week5 week6 week7 dose;

/*****************************************************************

  We use PROC CORR to calculate the estimates of Sigma, the assumed
  common covariance matrix, separately for each dose group.  The COV
  option asks for the covariance matrix to be printed.

*****************************************************************/

proc sort data=pigs1; by dose; run;
proc corr data=pigs1 cov; by dose;
  var week1 week3 week4 week5 week6 week7; run;

/*****************************************************************

  Use PROC GLM to carry out the multivariate analysis and profile
  analysis, respectively.  The description is exactly the same as
  for Example 1 (the dental study).  In the first call, we also show
  use of the MEANS statement to calculate the means for each dose
  group at each time.

*****************************************************************/

proc glm data=pigs1;
  class dose;
  model week1 week3 week4 week5 week6 week7 = dose / nouni;
  means dose;
  manova h=dose / printh printe;
run;

proc glm data=pigs1;
  class dose;
  model week1 week3 week4 week5 week6 week7 = dose / nouni;
  repeated week / printe nou;
run;
```

OUTPUT:

1

dose=1

The CORR Procedure

6 Variables: week1 week3 week4 week5 week6 week7

Covariance Matrix, DF = 4

	week1	week3	week4
week1	279.800000	158.550000	167.100000
week3	158.550000	1651.800000	1606.100000
week4	167.100000	1606.100000	1567.200000
week5	-34.800000	1625.200000	1592.900000
week6	476.950000	1972.950000	2010.900000
week7	252.500000	2076.250000	2077.500000

Covariance Matrix, DF = 4

	week5	week6	week7
week1	-34.800000	476.950000	252.500000
week3	1625.200000	1972.950000	2076.250000
week4	1592.900000	2010.900000	2077.500000
week5	1835.300000	2081.550000	2251.750000
week6	2081.550000	4472.800000	3989.000000
week7	2251.750000	3989.000000	3821.500000

Simple Statistics

Variable	N	Mean	Std Dev	Sum	Minimum	Maximum
week1	5	466.40000	16.72722	2332	445.00000	485.00000
week3	5	519.40000	40.64234	2597	460.00000	565.00000
week4	5	568.80000	39.58788	2844	510.00000	610.00000
week5	5	561.60000	42.84040	2808	504.00000	597.00000
week6	5	546.60000	66.87900	2733	436.00000	611.00000
week7	5	572.00000	61.81828	2860	466.00000	619.00000

Pearson Correlation Coefficients, N = 5
Prob > |r| under H0: Rho=0

	week1	week3	week4	week5	week6	week7
week1	1.00000	0.23322 0.7058	0.25234 0.6822	-0.04856 0.9382	0.42634 0.4741	0.24419 0.6922
week3	0.23322 0.7058	1.00000	0.99823 <.0001	0.93341 0.0204	0.72585 0.1650	0.82639 0.0845
week4	0.25234 0.6822	0.99823 <.0001	1.00000	0.93923 0.0178	0.75952 0.1363	0.84891 0.0689

2

dose=1

The CORR Procedure

Pearson Correlation Coefficients, N = 5
Prob > |r| under H0: Rho=0

	week1	week3	week4	week5	week6	week7
week5	-0.04856 0.9382	0.93341 0.0204	0.93923 0.0178	1.00000	0.72651 0.1645	0.85026 0.0680
week6	0.42634 0.4741	0.72585 0.1650	0.75952 0.1363	0.72651 0.1645	1.00000	0.96484 0.0079
week7	0.24419 0.6922	0.82639 0.0845	0.84891 0.0689	0.85026 0.0680	0.96484 0.0079	1.00000

3

dose=2

The CORR Procedure

6 Variables: week1 week3 week4 week5 week6 week7

Covariance Matrix, DF = 4

	week1	week3	week4
week1	1018.300000	1270.750000	738.900000
week3	1270.750000	1755.000000	998.500000
week4	738.900000	998.500000	783.700000
week5	1450.500000	2182.500000	1654.250000
week6	769.750000	1105.000000	1298.000000
week7	1232.500000	1978.750000	1430.750000

Covariance Matrix, DF = 4

	week5	week6	week7
week1	1450.500000	769.750000	1232.500000
week3	2182.500000	1105.000000	1978.750000
week4	1654.250000	1298.000000	1430.750000
week5	3851.500000	2800.750000	3519.500000
week6	2800.750000	2841.500000	2394.000000
week7	3519.500000	2394.000000	3312.000000

Simple Statistics

Variable	N	Mean	Std Dev	Sum	Minimum	Maximum
week1	5	494.40000	31.91081	2472	440.00000	520.00000
week3	5	551.00000	41.89272	2755	480.00000	590.00000
week4	5	574.20000	27.99464	2871	536.00000	610.00000
week5	5	567.00000	62.06045	2835	484.00000	637.00000
week6	5	603.00000	53.30572	3015	552.00000	671.00000
week7	5	644.00000	57.54998	3220	569.00000	702.00000

Pearson Correlation Coefficients, N = 5
Prob > |r| under H0: Rho=0

	week1	week3	week4	week5	week6	week7
week1	1.00000	0.95057 0.0131	0.82713 0.0840	0.73243 0.1593	0.45252 0.4442	0.67113 0.2149
week3	0.95057 0.0131	1.00000	0.85140 0.0672	0.83946 0.0753	0.49482 0.3967	0.82074 0.0886
week4	0.82713 0.0840	0.85140 0.0672	1.00000	0.95216 0.0125	0.86981 0.0553	0.88806 0.0442

---------- dose=2 ----------

The CORR Procedure

Pearson Correlation Coefficients, N = 5
Prob > |r| under H0: Rho=0

	week1	week3	week4	week5	week6	week7
week5	0.73243 0.1593	0.83946 0.0753	0.95216 0.0125	1.00000	0.84661 0.0704	0.98542 0.0021
week6	0.45252 0.4442	0.49482 0.3967	0.86981 0.0553	0.84661 0.0704	1.00000	0.78038 0.1194
week7	0.67113 0.2149	0.82074 0.0886	0.88806 0.0442	0.98542 0.0021	0.78038 0.1194	1.00000

---------- dose=3 ----------

The CORR Procedure

6 Variables: week1 week3 week4 week5 week6 week7

Covariance Matrix, DF = 4

	week1	week3	week4
week1	822.200000	705.400000	298.950000
week3	705.400000	885.800000	718.650000
week4	298.950000	718.650000	897.200000
week5	712.700000	1061.400000	1022.200000
week6	930.800000	1180.600000	1013.050000
week7	632.050000	953.850000	916.050000

Covariance Matrix, DF = 4

	week5	week6	week7
week1	712.700000	930.800000	632.050000
week3	1061.400000	1180.600000	953.850000
week4	1022.200000	1013.050000	916.050000
week5	1539.700000	1674.300000	1385.050000
week6	1674.300000	1910.200000	1493.450000
week7	1385.050000	1493.450000	1251.200000

Simple Statistics

Variable	N	Mean	Std Dev	Sum	Minimum	Maximum
week1	5	497.80000	28.67403	2489	472.00000	545.00000
week3	5	534.60000	29.76239	2673	498.00000	565.00000
week4	5	579.80000	29.95330	2899	540.00000	622.00000
week5	5	571.80000	39.23901	2859	524.00000	622.00000
week6	5	588.20000	43.70583	2941	532.00000	633.00000
week7	5	623.20000	35.37231	3116	583.00000	670.00000

Pearson Correlation Coefficients, N = 5
Prob > |r| under H0: Rho=0

	week1	week3	week4	week5	week6	week7
week1	1.00000	0.82657 0.0844	0.34807 0.5659	0.63343 0.2513	0.74273 0.1505	0.62316 0.2614
week3	0.82657 0.0844	1.00000	0.80613 0.0994	0.90885 0.0326	0.90760 0.0332	0.90604 0.0341
week4	0.34807 0.5659	0.80613 0.0994	1.00000	0.86971 0.0553	0.77383 0.1246	0.86459 0.0586

---------- dose=3 ----------

The CORR Procedure

Pearson Correlation Coefficients, N = 5
Prob > |r| under H0: Rho=0

	week1	week3	week4	week5	week6	week7
week5	0.63343 0.2513	0.90885 0.0326	0.86971 0.0553	1.00000	0.97628 0.0044	0.99789 0.0001
week6	0.74273 0.1505	0.90760 0.0332	0.77383 0.1246	0.97628 0.0044	1.00000	0.96602 0.0075
week7	0.62316 0.2614	0.90604 0.0341	0.86459 0.0586	0.99789 0.0001	0.96602 0.0075	1.00000

The GLM Procedure

Class Level Information

Class	Levels	Values
dose	3	1 2 3

Number of observations 15

The GLM Procedure

Level of		week1		week3	
dose	N	Mean	Std Dev	Mean	Std Dev
1	5	466.400000	16.7272233	519.400000	40.6423425
2	5	494.400000	31.9108132	551.000000	41.8927201
3	5	497.800000	28.6740301	534.600000	29.7623924

Level of		week4		week5	
dose	N	Mean	Std Dev	Mean	Std Dev
1	5	568.800000	39.5878769	561.600000	42.8404015
2	5	574.200000	27.9946423	567.000000	62.0604544
3	5	579.800000	29.9532970	571.800000	39.2390112

Level of		week6		week7	
dose	N	Mean	Std Dev	Mean	Std Dev

1	5	546.600000	66.8789952	572.000000	61.8182821
2	5	603.000000	53.3057220	644.000000	57.5499783
3	5	588.200000	43.7058349	623.200000	35.3723056

9

The GLM Procedure
Multivariate Analysis of Variance

E = Error SSCP Matrix

	week1	week3	week4
week1	8481.2	8538.8	4819.8
week3	8538.8	17170.4	13293
week4	4819.8	13293	12992.4
week5	8513.6	19476.4	17077.4
week6	8710	17034.2	17287.8
week7	8468.2	20035.4	17697.2

E = Error SSCP Matrix

	week5	week6	week7
week1	8513.6	8710	8468.2
week3	19476.4	17034.2	20035.4
week4	17077.4	17287.8	17697.2
week5	28906	26226.4	28625.2
week6	26226.4	36898	31505.8
week7	28625.2	31505.8	33538.8

Partial Correlation Coefficients from the Error SSCP Matrix / Prob > |r|

DF = 12	week1	week3	week4	week5	week6	week7
week1	1.000000	0.707584 0.0068	0.459151 0.1145	0.543739 0.0548	0.492366 0.0874	0.502098 0.0804
week3	0.707584 0.0068	1.000000	0.889996 <.0001	0.874228 <.0001	0.676753 0.0111	0.834899 0.0004
week4	0.459151 0.1145	0.889996 <.0001	1.000000	0.881217 <.0001	0.789575 0.0013	0.847786 0.0003
week5	0.543739 0.0548	0.874228 <.0001	0.881217 <.0001	1.000000	0.803051 0.0009	0.919350 <.0001
week6	0.492366 0.0874	0.676753 0.0111	0.789575 0.0013	0.803051 0.0009	1.000000	0.895603 <.0001
week7	0.502098 0.0804	0.834899 0.0004	0.847786 0.0003	0.919350 <.0001	0.895603 <.0001	1.000000

10

The GLM Procedure
Multivariate Analysis of Variance

H = Type III SSCP Matrix for dose

	week1	week3	week4
week1	2969.2	2177.2	859.4
week3	2177.2	2497.6	410
week4	859.4	410	302.53333333
week5	813	411.6	280.4
week6	4725.2	4428.8	1132.1333333
week7	5921.6	5657.6	1392.5333333

H = Type III SSCP Matrix for dose

	week5	week6	week7
week1	813	4725.2	5921.6
week3	411.6	4428.8	5657.6
week4	280.4	1132.1333333	1392.5333333
week5	260.4	1096.4	1352
week6	1096.4	8550.9333333	10830.933333
week7	1352	10830.933333	13730.133333

Characteristic Roots and Vectors of: E Inverse * H, where
H = Type III SSCP Matrix for dose
E = Error SSCP Matrix

Characteristic Root	Percent	Characteristic Vector V'EV=1 week1	week3	week4	week5
		week6	week7		

```
    2.76663572    57.81    0.01008494   -0.00856690    0.00598260   -0.01350074
                          -0.00631967    0.01895546
    2.01931265    42.19    0.02377927   -0.04047800    0.03355915    0.00129118
                          -0.01481413    0.01295337
    0.00000000     0.00   -0.00022690   -0.00372379   -0.01380715    0.01173179
                          -0.00015021    0.00199588
    0.00000000     0.00   -0.00425334    0.00094691    0.00882637   -0.00027390
                          -0.00381939    0.00358891
    0.00000000     0.00   -0.00592948   -0.00835257    0.00451460   -0.00286298
                          -0.00450358    0.00937569
    0.00000000     0.00   -0.00257775   -0.00142122    0.00128210   -0.00084350
                           0.01035699   -0.00651966
```

11

The GLM Procedure
Multivariate Analysis of Variance

MANOVA Test Criteria and F Approximations for
the Hypothesis of No Overall dose Effect
H = Type III SSCP Matrix for dose
E = Error SSCP Matrix

S=2 M=1.5 N=2.5

Statistic	Value	F Value	Num DF	Den DF	Pr > F
Wilks' Lambda	0.08793025	2.77	12	14	0.0363
Pillai's Trace	1.40330988	3.14	12	16	0.0176
Hotelling-Lawley Trace	4.78594837	2.63	12	8.2712	0.0852
Roy's Greatest Root	2.76663572	3.69	6	8	0.0464

NOTE: F Statistic for Roy's Greatest Root is an upper bound.
NOTE: F Statistic for Wilks' Lambda is exact.

12

The GLM Procedure

Class Level Information

Class	Levels	Values
dose	3	1 2 3

Number of observations 15

13

The GLM Procedure
Repeated Measures Analysis of Variance

Repeated Measures Level Information

Dependent Variable	week1	week3	week4	week5	week6	week7
Level of week	1	2	3	4	5	6

Partial Correlation Coefficients from the Error SSCP Matrix / Prob > |r|

DF = 12	week1	week3	week4	week5	week6	week7
week1	1.000000	0.707584 0.0068	0.459151 0.1145	0.543739 0.0548	0.492366 0.0874	0.502098 0.0804
week3	0.707584 0.0068	1.000000	0.889996 <.0001	0.874228 <.0001	0.676753 0.0111	0.834899 0.0004
week4	0.459151 0.1145	0.889996 <.0001	1.000000	0.881217 <.0001	0.789575 0.0013	0.847786 0.0003
week5	0.543739 0.0548	0.874228 <.0001	0.881217 <.0001	1.000000	0.803051 0.0009	0.919350 <.0001
week6	0.492366 0.0874	0.676753 0.0111	0.789575 0.0013	0.803051 0.0009	1.000000	0.895603 <.0001
week7	0.502098 0.0804	0.834899 0.0004	0.847786 0.0003	0.919350 <.0001	0.895603 <.0001	1.000000

E = Error SSCP Matrix

week_N represents the contrast between the nth level of week and the last

week_1	week_2	week_3	week_4	week_5

week_1	25083.6	13574.0	12193.2	4959.0	2274.8
week_2	13574.0	10638.4	9099.2	4354.6	-968.2
week_3	12193.2	9099.2	11136.8	4293.8	1623.6
week_4	4959.0	4354.6	4293.8	5194.4	-365.8
week_5	2274.8	-968.2	1623.6	-365.8	7425.2

14

The GLM Procedure
Repeated Measures Analysis of Variance

Partial Correlation Coefficients from the Error SSCP Matrix of the Variables Defined by the Specified Transformation / Prob > |r|

DF = 12	week_1	week_2	week_3	week_4	week_5
week_1	1.000000	0.830950 0.0004	0.729529 0.0047	0.434442 0.1380	0.166684 0.5863
week_2	0.830950 0.0004	1.000000	0.835959 0.0004	0.585791 0.0354	-0.108936 0.7231
week_3	0.729529 0.0047	0.835959 0.0004	1.000000	0.564539 0.0444	0.178544 0.5595
week_4	0.434442 0.1380	0.585791 0.0354	0.564539 0.0444	1.000000	-0.058901 0.8484
week_5	0.166684 0.5863	-0.108936 0.7231	0.178544 0.5595	-0.058901 0.8484	1.000000

Sphericity Tests

Variables	DF	Mauchly's Criterion	Chi-Square	Pr > ChiSq
Transformed Variates	14	0.0160527	41.731963	0.0001
Orthogonal Components	14	0.0544835	29.389556	0.0093

Manova Test Criteria and Exact F Statistics for the Hypothesis of no week Effect
H = Type III SSCP Matrix for week
E = Error SSCP Matrix

S=1 M=1.5 N=3

Statistic	Value	F Value	Num DF	Den DF	Pr > F
Wilks' Lambda	0.03881848	39.62	5	8	<.0001
Pillai's Trace	0.96118152	39.62	5	8	<.0001
Hotelling-Lawley Trace	24.76092347	39.62	5	8	<.0001
Roy's Greatest Root	24.76092347	39.62	5	8	<.0001

15

The GLM Procedure
Repeated Measures Analysis of Variance

Manova Test Criteria and F Approximations
for the Hypothesis of no week*dose Effect
H = Type III SSCP Matrix for week*dose
E = Error SSCP Matrix

S=2 M=1 N=3

Statistic	Value	F Value	Num DF	Den DF	Pr > F
Wilks' Lambda	0.17905151	2.18	10	16	0.0793
Pillai's Trace	1.07058517	2.07	10	18	0.0856
Hotelling-Lawley Trace	3.19076786	2.42	10	9.6	0.0937
Roy's Greatest Root	2.66824588	4.80	5	9	0.0205

NOTE: F Statistic for Roy's Greatest Root is an upper bound.
NOTE: F Statistic for Wilks' Lambda is exact.

16

The GLM Procedure
Repeated Measures Analysis of Variance
Tests of Hypotheses for Between Subjects Effects

Source	DF	Type III SS	Mean Square	F Value	Pr > F
dose	2	18548.0667	9274.0333	1.06	0.3782
Error	12	105434.2000	8786.1833		