Stat 771, Fall 2011: Homework 2
Due Wednesday, February 6

1. Let
$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \text{ and } \mathbf{c} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

   (a) Find $3\mathbf{B}$.
   Answer:
$$3\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}.$$

   (b) Find $\mathbf{A} - \mathbf{B}$.
   Answer:
$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

   (c) Find $\mathbf{AB}$.
   Answer:
$$\begin{aligned} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} &= \begin{bmatrix} \begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} & \begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 2\times 2 + 1\times(-1) & 2\times(-1) + 1\times 2 \\ 1\times 2 + 2\times(-1) & 1\times(-1) + 2\times 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \end{aligned}$$

   Note that you only need to multiply one matrix by $\frac{1}{3}$ to get the inverse of the other.

   (d) Find $|\mathbf{A}|$.
   Answer: $2\times 2 - 1\times 1 = 3$.

   (e) Find $\mathbf{A}^{-1}$.
   Answer: From the formula
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$
   we get
$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

   (f) Is $\mathbf{A}$ full rank? Why or why not?
   Answer: Yes, because either (a) $\mathbf{A}^{-1}$ exists, or (b) $|\mathbf{A}| \neq 0$.

   (g) Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Show
$$\mathbf{x}'\mathbf{A}\mathbf{x} = 2(x_1^2 + x_1x_2 + x_2^2).$$

   Answer:
$$\begin{aligned} \mathbf{x}'\mathbf{A}\mathbf{x} &= \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} \\ &= 2x_1^2 + x_1x_2 + x_2x_1 + 2x_2^2 \\ &= 2(x_1^2 + x_1x_2 + x_2^2). \end{aligned}$$

   (h) Use $\mathbf{A}^{-1}$ to solve the system of two equations in two unknowns
$$\left\{ \begin{array}{ccccc} 2x_1 & + & x_2 & = & 1 \\ x_1 & + & 2x_2 & = & 3 \end{array} \right\}.$$

That is, solve the system

$$\mathbf{Ax} = \mathbf{c}.$$

Answer:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} - 1 \\ -\frac{1}{3} + 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{3} \end{bmatrix}.$$

(i) Show $tr(\mathbf{A} + \mathbf{B}) = tr(\mathbf{A}) + tr(\mathbf{B})$.

Answer: $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ so $tr(\mathbf{A} + \mathbf{B}) = 8$. Also, $tr(\mathbf{A}) = 4$ and $tr(\mathbf{B}) = 4$ so the result follows.

(j) Let $\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ be a *random vector* with mean $\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ and covariance matrix $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$. Use results on p. 46 to find $E(\mathbf{AY} + \mathbf{c})$ and $var(\mathbf{AY} + \mathbf{c})$.

2. Say Y_{ij} is the j^{th} measurement on subject i, where $i = 1, \ldots, n$ and $j = 1, \ldots, 4$. All n individuals have the same mean vector (multivariate one-sample problem). Define

$$\mathbf{Y}_i = \begin{bmatrix} Y_{i1} \\ Y_{i2} \\ Y_{i3} \\ Y_{i4} \end{bmatrix} \text{ and } E(\mathbf{Y}_i) = \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}.$$

Say we want to show that the mean changes over time. The null hypothesis is that *this doesn't happen* $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$. Find a 3×4 matrix $\mathbf{C}$ such that

$$\mathbf{C}\boldsymbol{\mu} = \mathbf{0} \Leftrightarrow \mu_1 = \mu_2 = \mu_3 = \mu_4.$$

Hint: all four means are equal if and only if three pairs of means are *simultaneously* equal.
Answer: There are many possible matrices that enforce this; here's one:

$$\mathbf{C} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Then $\mathbf{C}\boldsymbol{\mu} = \mathbf{0}$ implies that $\mu_1 = \mu_2$, $\mu_2 = \mu_3$, and $\mu_3 = \mu_4$ simultaneously hold.

3. Let

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \; var(\mathbf{Y}) = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}, \text{ and } \mathbf{c} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}.$$

Use $var(c_1Y_1 + c_2Y_2) = c_1^2 var(Y_1) + c_2^2 var(Y_2) + 2c_1c_2 cov(Y_1, Y_2)$ to show

$$var(\mathbf{cY}) = \mathbf{c\Sigma c}'.$$

Answer:

$$\begin{aligned} \mathbf{c\Sigma c}' &= \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 c_1 + \sigma_{12} c_2 \\ \sigma_{12} c_1 + \sigma_2^2 c_2 \end{bmatrix} \\ &= \sigma_1^2 c_1^2 + \sigma_{12} c_2 c_1 + \sigma_{12} c_1 c_2 + \sigma_2^2 c_2^2 \\ &= \sigma_1^2 c_1^2 + 2\sigma_{12} c_1 c_2 + +\sigma_2^2 c_2^2 \\ &= c_1^2 var(Y_1) + c_2^2 var(Y_2) + 2c_1c_2 cov(Y_1, Y_2) \end{aligned}$$

4. Consider the dental data of Example 1 (pp. 3–4 in the notes). Let Y_{ij} be the jth measurement on child i, where $i = 1, \ldots, 27$. Let $a_1 = 8$, $a_2 = 10$, $a_3 = 12$, and $a_4 = 14$ be the ages at the four measurements. Let $x_i = 0$ if child i is a girl and $x_i = 1$ if a boy.

   (a) Obtain profile plots (or "spaghetti" plots) of the boys and girls separately, but on the same scale. Do there appear to be differences between boys and girls? Elaborate. I posted some sample SAS code to aid you in this.
   Answer: Yes, in general boys have larger measurements than girls, reflected in the estimated mean for each group.

   (b) In your favorite SAS procedure (e.g. GLM or REG, GLM would be easier here) or some other package, fit the regression model

$$Y_{ij} = \beta_{00} + \beta_{01}x_i + \beta_{10}a_j + \beta_{11}a_j x_i + e_{ij}.$$

   Think about this model; in words what does it assume? Draw a hypothetical mean functions versus age for boys and girls separately.
   Answer: Boys and girls each have a unique line with possibly different slopes and intercepts.

   Formally test that boys and girls have the exact same mean growth over time $H_0 : \beta_{01} = \beta_{11} = 0$. Formally test that boys and girls grow at the same rate, but have different intercepts $H_0 : \beta_{11} = 0$.
   Answer: The following SAS code

```
proc glm data=dental;
 model distance=age g age*g / solution;
 contrast 'H0: no difference' age*g 1, g 1;
```

   gives

Contrast	DF	Contrast SS	Mean Square	F Value	Pr > F
H0: no difference	2	152.5790088	76.2895044	14.98	<.0001

Source	DF	Type III SS	Mean Square	F Value	Pr > F
age*g	1	12.11415194	12.11415194	2.38	0.1261

   We reject that boys and girls have the same growth over time. However, we accept that we can drop the interaction term, i.e. we accept that boys and girls grow at the same rate (under the model).

   An assumption going into this model is that $e_{ij} \overset{iid}{\sim} N(0, \sigma^2)$; this will be relaxed later on when we allow $\mathbf{e}_i = (e_{i1}, e_{i2}, e_{i3}, e_{i4})'$ to be *correlated*.

   (c) Now let's **not** assume that the overall population means follow a line, but rather are unstructured:

$$Y_{ij} = \mu_j + \delta_j x_i + e_{ij}.$$

   Formally test no difference between boys and girls $H_0 : \delta_1 = \delta_2 = \delta_3 = \delta_4 = 0$. This is very easily carried out in SAS PROC GLM using the CLASS statement for both GENDER and AGE, and the appropriate CONTRAST statement. It may help to rewrite the model as

$$Y_{ijk} = \mu_{ij} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk},$$

   where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are age effects and β_1, β_2 are gender effects. Then you are formally testing

$$H_0 : \left\{ \begin{array}{l} \mu_{11} - \mu_{12} = \beta_1 + (\alpha\beta)_{11} - \beta_2 - (\alpha\beta)_{12} = 0 \\ \mu_{21} - \mu_{22} = \beta_1 + (\alpha\beta)_{21} - \beta_2 - (\alpha\beta)_{22} = 0 \\ \mu_{31} - \mu_{32} = \beta_1 + (\alpha\beta)_{31} - \beta_2 - (\alpha\beta)_{32} = 0 \\ \mu_{41} - \mu_{42} = \beta_1 + (\alpha\beta)_{41} - \beta_2 - (\alpha\beta)_{42} = 0 \end{array} \right\}.$$

   My code looks like

```
contrast 'H0: no difference' gender 1 -1 age*gender 1 -1  0  0  0  0  0  0,
                             gender 1 -1 age*gender 0  0  1 -1  0  0  0  0,
                             gender 1 -1 age*gender 0  0  0  0  1 -1  0  0,
                             gender 1 -1 age*gender 0  0  0  0  0  0  1 -1;
```

   Answer: This gives the output

Contrast	DF	Contrast SS	Mean Square	F Value	Pr > F
H0: no difference	4	154.4573864	38.6143466	7.34	<.0001

   We reject that there is no different between boys and girls over time in the more general model.