Solutions for Stat 512 — Take home exam I

1. Let $Y$ has an exponential distribution with mean $\beta$. Prove that $W = \sqrt{Y}$ has a Weibull density. (Hint: Using the C.D.F technique) (20 pts)

Solution:

$$Y \sim \text{Exp}(\beta) \implies f_Y(y) = \frac{1}{\beta} e^{-\frac{y}{\beta}}, \ y \geq 0$$

Hence,

$$F_W(w) = P(W \leq w) = P\left(\sqrt{Y} \leq w\right) = P(0 < Y \leq w^2) = \int_0^{w^2} \frac{1}{\beta} e^{-\frac{y}{\beta}} dy = -e^{\left(-\frac{y}{\beta}\right)}\bigg|_{y=0}^{y=w^2} = 1 - e^{-\frac{w^2}{\beta}}, \ w \geq 0$$

Furthermore,

$$f_W(w) = \frac{dF_W(w)}{dw} = \frac{2w}{\beta} e^{-\frac{w^2}{\beta}}, \ w \geq 0$$

From Wikipedia, the pdf of Weibull distribution is:

$$f_Y(y) = \frac{k}{\lambda} \left(\frac{y}{\lambda}\right)^{k-1} e^{-\left(y/\lambda\right)^k}, \ y \geq 0$$

Hence, the pdf of $W$ is Weibull distribution with $k = 2$ and $\lambda = \sqrt{\beta}$.

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2. Let $Y_1, Y_2 \sim N(0, 1)$, $Y_1, Y_2$ are independent random variable. Find the distribution of $U = \frac{Y_1}{Y_2}$. (Hint: Using transformation technique) (20 pts)

Solution:

$$Y_1, Y_2 \overset{\text{ind.}}{\sim} N(0, 1)$$

$$\implies f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1)f_{Y_2}(y_2) = \frac{1}{2\pi} e^{\left(-\frac{y_1^2 + y_2^2}{2}\right)}, \ y_1, y_2 \in (-\infty, \infty)$$

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For part (a), let \( U = \frac{Y_1}{Y_2} \) and \( V = Y_2 \). Hence,

\[
\begin{align*}
Y_1 &= h_1^{-1}(U, V) = UV \\
Y_2 &= h_2^{-1}(U, V) = V
\end{align*}
\]

\[\implies J = \begin{vmatrix} V & U \\ 0 & 1 \end{vmatrix} = V\]

How about the support?
If \( V > 0 \), then \(-\infty < U < \infty\) and 

\[
f_{U,V}(u, v) = f_{Y_1,Y_2}(h_1^{-1}, h_2^{-1}) |J|
\]

\[= \frac{v}{2\pi} \exp\left(-\frac{u^2v^2 + v^2}{2}\right), \quad v \in (0,\infty), u \in (-\infty, \infty)\]

If \( V < 0 \), then \(-\infty < U < \infty\) and 

\[
f_{U,V}(u, v) = f_{Y_1,Y_2}(h_1^{-1}, h_2^{-1}) |J|
\]

\[= -\frac{v}{2\pi} \exp\left(-\frac{u^2v^2 + v^2}{2}\right), \quad v \in (-\infty, 0), u \in (-\infty, \infty)\]

If \( V = 0 \), then \( U \) is undefined. (But do we care about the case \( V = 0 \)?)

Hence,

\[
f_U(u) = \int_{-\infty}^{\infty} \frac{v}{2\pi} \exp\left(-\frac{u^2v^2 + v^2}{2}\right) dv + \int_{-\infty}^{0} -\frac{v}{2\pi} \exp\left(-\frac{u^2v^2 + v^2}{2}\right) dv
\]

For the first integral, let \( w = v^2 \), then

\[
\int_{0}^{\infty} \frac{v}{2\pi} \exp\left(-\frac{u^2v^2 + v^2}{2}\right) dv = \int_{0}^{\infty} \frac{1}{4\pi} \exp\left(-\frac{w(u^2 + 1)}{2}\right) dw
\]

\[= \frac{1}{4\pi} \left(-\frac{2}{u^2 + 1}\right) \exp\left(-\frac{w}{2/(u^2 + 1)}\right) \bigg|_{0}^{\infty}
\]

\[= \frac{1}{2\pi} \frac{1}{u^2 + 1}, u \in (-\infty, \infty)\]

For the second integral, let \( w = v^2 \), then

\[
\int_{-\infty}^{0} -\frac{v}{2\pi} \exp\left(-\frac{u^2v^2 + v^2}{2}\right) dv = \int_{-\infty}^{0} -\frac{1}{4\pi} \exp\left(-\frac{w(u^2 + 1)}{2}\right) dw
\]

\[= -\frac{1}{4\pi} \left(-\frac{2}{u^2 + 1}\right) \exp\left(-\frac{w}{2/(u^2 + 1)}\right) \bigg|_{0}^{\infty}
\]

\[= \frac{1}{2\pi} \frac{1}{u^2 + 1}, u \in (-\infty, \infty)\]
Therefore,

\[ f_U(u) = \frac{1}{2\pi} \frac{1}{u^2 + 1} + \frac{1}{2\pi} \frac{1}{u^2 + 1} = \frac{1}{\pi} \frac{1}{u^2 + 1}, \quad u \in (-\infty, \infty) \]

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3. Suppose that a unit of mineral ore contains a proportion \( Y_1 \) of metal A and a proportion \( Y_2 \) of metal B. Experience has shown that the joint probability density function of \( Y_1 \) and \( Y_2 \) is uniform over the region \( 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, 0 \leq y_1 + y_2 \leq 1 \). Let \( U = Y_1 + Y_2 \), the proportion of either metal A or B per unit. Find

a) the probability density function for \( U \). \( \text{ (10 pts)} \)

b) \( E(U) \) and \( Var(U) \) by using the answer to part (a). \( \text{ (10 pts)} \)

Solution:
For part (a): Firstly we should find out the joint density of \( (Y_1, Y_2) \), it is shown as below:

\[ f_{Y_1,Y_2}(y_1, y_2) = 2 \quad if \quad 0 \leq y_1, y_2 \leq 1, \quad 0 \leq y_1 + y_2 \leq 1 \]

Notice: The joint density is not 1 since we have an additional constrain: \( 0 \leq y_1 + y_2 \leq 1 \). The region of \( (Y_1, Y_2) \) that satisfies the condition \( U \leq u \) is shown below:
Hence,

\[ F_U(u) = P(U \leq u) \]
\[ = P(Y_1 + Y_2 \leq u) \]
\[ = \int_0^u \int_0^{u-y_1} 2dy_2dy_1 \]
\[ = u^2, \quad u \in [0, 1] \]

Otherwise \( F_U(u) = 0 \) for \( u \in (-\infty, 0) \) and 1 for \( u \in (1, \infty) \). Also, \( f_U(u) = \frac{dF_U(u)}{du} = 2u \) for \( u \in [0, 1] \).

For part (b),

\[ E(U) = \int_0^1 u \cdot 2udu = \frac{2}{3} \]
\[ E(U^2) = \int_0^1 u^2 \cdot 2udu = \frac{1}{2} \]

Hence, \( E(U) = \frac{2}{3} \) and \( Var(U) = E(U^2) - [E(U)]^2 = \frac{1}{18} \).
4. The total time from arrival to completion of service at a fast-food outlet, $Y_1$ and the time spent waiting in line before arriving at the service window, $Y_2$, have a joint density function:

$$f(y_1, y_2) = \begin{cases} 
e^{-y_1}, & 0 \leq y_2 \leq y_1 < \infty \\ 0, & \text{otherwise} \end{cases}$$

Another random variable of interest is $U = Y_1 - Y_2$, the time spent at the service window. Find:

(a) the probability density function for $U$ using bivariate transformation technique. (10 pts)

(b) $E(U)$ and $Var(U)$ by using the answer to part (a). (10 pts)

**Solution:**

For part (a), we can set up another r.v $V$ such that $V = Y_1$. Hence,

$$\begin{align*}
Y_1 &= V \\
Y_2 &= V - U \\
\implies J &= \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = 1
\end{align*}$$

It is not hard to identify the support of $(U, V)$ is: $U \geq 0$, $V \geq U$. Hence,

$$f_{U,V}(u, v) = e^{-v} \cdot |1| = e^{-v}, \quad u \geq 0, \quad v \geq u$$

And,

$$f_U(u) = \int_u^\infty e^{-v} \, dv = e^{-u}, \quad u \geq 0$$

Hence, the time spent at the service window, $U$, has standard Exponential distribution.

For part (b), from the property of Exponential distribution, the mean and variance for $U$ are all 1.

5. Let $Y_1, \ldots, Y_n$ be i.i.d random variables such that for $0 < p < 1$, $P(Y_i = 1) = p$ and $P(Y_i = 0) = q = 1 - p$. (Such random variables are called Bernoulli random variables.)

(a) Find the moment generating function for the Bernoulli random variable $Y_1$. Make sure you show your steps. (10pts)

(b) Find the moment generating function for $W = Y_1 + Y_2 + \ldots + Y_n$. Can you recognize which known distribution has this mgf? (10 pts)
Solution:

a. For Bernoulli distribution:

\[ m_{Y_1}(t) = E(e^{tY_1}) = \sum_{y_1=0\ or\ 1} e^{ty_1} p^{y_1} (1-p)^{1-y_1} = (1-p) + pe^t \]

b. Since \( Y_1, \ldots, Y_n \) are i.i.d,

\[ m_W(t) = [m_{Y_1}(t)]^n = [(1-p) + pe^t]^n \]

Hence, the distribution of \( W \) is Binomial\((n, p)\).

Extra credit question: If \( Y_i, i = 1, 2, \) are independent Gamma\((\alpha_i, 1)\) random variables, find the marginal distributions of \( U_1 = \frac{Y_1}{Y_1 + Y_2} \) and \( U_2 = \frac{Y_2}{Y_1 + Y_2} \).

Solution:

It is easier to find the distribution of \( U_1 \) and \( U_2 \) separately. For \( U_1 \),

\[ U = \frac{Y_1}{Y_1 + Y_2} \implies \begin{cases} Y_1 = UV \\ Y_2 = V(1-U) \end{cases} \implies J = V > 0 \quad (since\ gamma) \]

For the support:

\[ \begin{cases} Y_1 = UV > 0 \\ Y_2 = V(1-U) > 0 \end{cases} \implies \begin{cases} V > 0 \\ 0 < U < 1 \end{cases} \]

Hence,

\[ f_{U,V}(u,v) = \frac{1}{\Gamma(\alpha_1)} (uv)^{\alpha_1-1} e^{-uv} \frac{1}{\Gamma(\alpha_2)} (v(1-u))^{\alpha_2-1} e^{-v(1-u)} \cdot v \]

\[ = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{\alpha_1-1}(1-u)^{\alpha_2-1} v^{\alpha_1+\alpha_2-1} e^{-v} \]

From above joint density, it is straightforward to see that \( U \) and \( V \) are independent. Also the kernel of \( U \) is \( u^{\alpha_1-1}(1-u)^{\alpha_2-1} \) which is the kernel for \( Beta(\alpha_1, \alpha_2) \). Hence, \( U_1 = \frac{Y_1}{Y_1 + Y_2} \) has \( Beta(\alpha_1, \alpha_2) \) distribution. (You can also integrate the joint density w.r.t \( v \) to get the marginal density of \( U \)). Similarly, \( U_2 \) has \( Beta(\alpha_2, \alpha_1) \) distribution.