Homework 04 Solution

STAT 509 Statistics for Engineers
Summer 2017 Section 001
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Question 01

Answer following questions about the random experiment: toss 2 dice:

a What is size of the sample space of the random experiment?

b Define random variable X = sum of the numbers. What is the range of X?

c Define random variable Y = sum of the odd numbers. What is the range of Y?

d Define random variable Z = number of sixes. What are the possible values of Z?

In conclusion, for the one random experiment, we have different range coming from different random variable (depending on what our interest is)!

Solution:
(a) S = $\{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 26, \ldots, 61, 62, 63, 64, 65, 66\}$. Total $6 \times 6 = 36$ elements.
(b) $X \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.
(c) $Y \in \{2, 4, 6, 8, 10\}$.
(d) $Z \in \{0, 1, 2\}$.

Question 02

What is the probability mass function (pmf) of the random variable Z in problem 1(d)? Try to plot the pmf using R. (Reference R code in notes chapter 3 page 13).

Solution:

$$
\begin{aligned}
P(Z=0) =& P(\text{Zero sixes}) \\
=& P(\text{No six in the first dice and No six in the second dice}) \\
=& \frac{5\times 5}{36} = \frac{25}{36} \\
P(Z=1) =& P(\text{Six in either of the dice}) \\
=& \frac{(1\times 5)+(5\times 1)}{36} \\
=& \frac{10}{36} \\
P(Z=2)P(\text{Six in both dices}) =& \frac{1}{36}
\end{aligned}
$$

So the pmf can be written as:

Z	0	1	2
P(Z)	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

```
## PMF plot code
z <- c(0,1,2)
pmf <- c(25/36, 10/36, 1/36)
plot(z, pmf, type="h", xlab="z", ylab="PMF")
abline(h=0)
```

Question 03

What is the cumulative distribution function (cdf) of the random variable Z in problem 1(d)? Try to plot the cdf using R. (Reference R code in notes chapter 3 page 13).

Solution:

cdf:

Z	0	1	2
P(Z)	$\frac{25}{36}$	$\frac{10}{36}+\frac{25}{36}=\frac{35}{36}$	$\frac{10}{36}+\frac{25}{36}+\frac{1}{36}=1$

```
## CDF plot code
cdf <- c(0, cumsum(pmf))
cdf.plot <- stepfun(z, cdf, f=0)
plot.stepfun(cdf.plot, xlab="z", ylab="CDF",
verticals=FALSE, main="", pch=16)
```

Question 04

The range of the random variable X is $\{0, 1, 2, 3, a\}$, in which a is unknown. If each outcome is equally likely and the mean (expectation) of X is 3, what is the value of a?

Solution:

Equally likely outcome means $P(X = x) = 1/5\ \forall x \in \{0, 1, 2, 3, a\}$. So the expected value of X is,

$$\begin{aligned}
E(X) =& \frac{0+1+2+3+a}{5} = \frac{6+a}{5} \\
E(X) =& 3 \\
\frac{6+a}{5} =& 3 \\
6+a =& 15 \\
a =& 15-6 = 9
\end{aligned}$$

Question 05

Suppose that the random variable X has the pmf:

$$P_X(x) = \frac{1}{12}(c - 2x)$$

with range $\{0, 1, 2\}$.

a Find the value of c.

b Compute the expectation and variance of Y.

Solution:

We know $\sum_x P_X(x) = 1$. We find,

$$\begin{aligned}
&\sum_x P_X(x) \\
=&\sum_x \frac{1}{12}(c-2x) \\
=&\frac{1}{12}\sum_x (c-2x) \\
=&\frac{1}{12}(\sum_x c - 2\sum_x x) \\
=&\frac{1}{12}(3c-6) \\
=&\frac{1}{4}(c-2)
\end{aligned}$$

$$\begin{aligned}
\sum_x P_X(x) =& 1 \\
\frac{1}{4}(c-2) =& 1 \\
c-2 =& 4 \\
c =& 6.
\end{aligned}$$

$$P_X(x) = \frac{1}{12}(6-2x) = \frac{1}{6}(3-x)$$

Expectation:

$$\begin{aligned}
E(X) =& \sum_x xP_X(x) \\
=& \sum_x \frac{1}{6}x(3-x) \\
=& \frac{1}{6}\sum_x x(3-x) \\
=& \frac{1}{6}(0 + 1(3-1) + 2(3-2)) \\
=& \frac{4}{6}.
\end{aligned}$$

Variance:

$$\begin{aligned} E(X^2) &= \sum_x x^2 P_X(x) \\ &= \sum_x \frac{1}{6} x^2 (3-x) \\ &= \frac{1}{6} \sum_x x^2 (3-x) \\ &= \frac{1}{6}(0 + 1(3-1) + 4(3-2)) \\ &= \frac{6}{6} = 1. \\ V(X) &= E(X^2) - (E(X))^2 \\ &= 1 - (\frac{4}{6})^2 = 0.56. \end{aligned}$$

Question 06

A pencil company has four extruders for making pencil lead. The maintenance manager has determined from historical data that the number of extruders to go down (out of operation) on any given day is as follows: 0 extruder, 50%; 1 extruder, 30%; 2 extruders, 10%; 3 extruders, 5%; 4 extruders, 5%.

a Find the probability that 3 or more extruders are down.

b What is the expected number of extruders down?

c Find the variance for the number of extruders down.

d Find the standard deviation for the number of extruders down.

Solution:

Let X = Number of extruders down. So the pmf is:

X	0	1	2	3	4
P(X)	0.50	0.30	.10	0.05	0.05

(a) P(3 or more extruders down) = $P(X \geq 3) = P(X = 3) + P(X = 4) = 0.10$.

(b) $E(X) = \sum_x x P_X(x)$
$= (0 \times 0.5) + (1 \times 0.3) + (2 \times 0.1) + (3 \times 0.05) + (4 \times 0.05)$
$= 0.85$.

(c)

$$\begin{aligned} E(X^2) &= \sum_x x^2 P_X(x) \\ &= (0 \times 0.5) + (1 \times 0.3) + (4 \times 0.1) + (9 \times 0.05) + (16 \times 0.05) \\ &= 2.45 \\ V(X) &= E(X^2) - (E(X))^2 \\ &= 2.45 - 0.7225 \\ &= 1.7275 \end{aligned}$$

Standard Deviation = $\sqrt{Var(X)} = 1.314344$