Homework 08 Solution

STAT 509 Statistics for Engineers
Summer 2017 Section 001
Instructor: Tahmidul Islam

1. The number of calls received by a telephone answering service follows a Poisson distribution. The calls average 20 per hour.

   (a) What is the probability of waiting more than 15 minutes between two calls? Use both Poisson and Exponential distribution to find the answer. (Hint: 15 minutes = 0.25 hour).

   Solution:
   Let, X = Number of calls received per hour. $X \sim Pois(\lambda = 20)$. Let Y = Number of calls received in 15 minutes. $Y \sim Pois(\lambda = \frac{20 \times 15}{60} = 5)$.

$$P(Y = 0) = \frac{e^{-5}5^0}{0!} = 0.006737947.$$

   Let T = time between two calls. $T \sim \exp(\lambda = 20)$.

$$\begin{aligned} P(T > 15\ mins) = P(T > \frac{15}{60}\ hours) = \\ \exp(-20 \times \frac{15}{60}) = \exp(-5) \\ = 0.006737947. \end{aligned}$$

2. Suppose X has an exponential distribution with an expectation of 10. Calculate $P(X < 15|X > 10)$. (Hint: apply the lack of memory property).

   Solution:

   $E(X) = \frac{1}{\lambda} = 10$. So, $\lambda = 0.1$.

$$\begin{aligned} P(X < 15|X > 10) =& 1 - P(X > 15|X > 10) \\ =& 1 - P(X > 10 + 5|X > 10) \\ =& 1 - P(X > 5) \\ =& P(X < 5) \\ =& 1 - \exp(-0.1 \times 5) \\ =& 0.3934693. \end{aligned}$$

3. Explosive devices used in mining operations produce (nearly) circular craters when detonated. The radii of these craters, say, Y, follow an exponential distribution with $\lambda = 0.10$.

   (a) Find the proportion of radii that will exceed 20 meters.

   (b) Find the probability that a single denotation will produce a radius between 5 and 15 meters.

   (c) The area of the crater is $W = \pi Y^2$. Find the expected (mean) area produced by the explosive devices; that is, compute E(W).

Solution:

$Y \sim exp(0.10)$.

(a)

$$P(Y > 20) = \exp(-0.10\ 20) = 0.1353353.$$

(b)

$$\begin{aligned} P(5 < Y < 15) &= P(Y < 15) - P(Y < 5) \\ &= 1 - \exp(-.1\ 15) - (1 - \exp(-.1\ 5)) \\ &= \exp(-.1\ 5) - \exp(-.1\ 15) = 0.3834005. \end{aligned}$$

(c) $E(W) = E(\pi Y^2) = \pi E(Y^2)$.

$$\begin{aligned} Var(Y) =& E(Y^2) - (E(Y))^2 \\ E(Y^2) =& Var(Y) + (E(Y))^2 \\ =& \frac{1}{0.1^2} + (\frac{1}{0.1})^2 \\ =& 200. \\ E(W) =& \pi\ 200 = 628.3185\ meter^2. \end{aligned}$$

4. For a type of airplane, the time to maintenance, Y (measured in weeks), varies according to the following pdf:

$$f_Y(y) = ce^{-y/4};\ y > 0.$$

   (a) What is the value of c? (Hint: Is this an exponential distribution?)
   (b) Calculate $E(Y)$ and $E(Y^2)$.
   (c) Let t be a fixed constant. Show that, for $t < \frac{1}{4}$,

$$M_Y(t) = E(e^{tY}) = \frac{1}{1-4t}.$$

   Hint: $E(e^{tY}) = \int_0^\infty e^{ty} f_Y(y) dy$.
   (d) Find $M'_Y(t) = \frac{d}{dt} M_Y(t)$.
   (e) Find $M'_Y(0) = \frac{d}{dt} M_Y(t)|_{t=0}$. Does it match with E(Y) in part b?
   This function $M_Y(t) = E(e^{tY})$ is called the *moment generating function of Y*. How do you think you could calculate $E(Y^2)$ using the moment-generating function? How about $E(Y^3)$? How about $E(Y^k)$ any an arbitrary postive integer? (You might realize that $E(e^{tY})$ is basically the Laplace transform of the pdf $f_Y(y)$.

5. An article in *Financial Markets Institutions and Instruments* modeled average annual losses (in billions of dollars) of the Federal Deposit Insurance Corporation (FDIC) with a Weibull distribution with parameters $\delta = 1.9317$ and $\beta = 0.8472$. Use R to determine the following:

   (a) Probability of a loss greater than 2 billion.
   (b) Probability of a loss between 2 and 4 billion.
   (c) Mean and variance of loss. (Hint: R command for Gamma function is `gamma()`).