Homework 09 Solution

STAT 509 Statistics for Engineers
Summer 2017 Section 001
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1. Suppose Y is a normal random variable with a mean of 37 and a standard deviation of 4.25. Apply **Empirical rule**, calculate the following probability.

   (a) Approximately what proportion of the distribution is less than 32.75.

   (b) Approximately what proportion of the distribution is greater than 49.75.

   (c) Approximately what proportion of the distribution lies between 32.75 and 49.75.

   Solution:

   (a)

$$
\begin{aligned}
P(X < 32.75) =& P(\frac{X-\mu}{\sigma} < \frac{32.75-37}{4.25}) \\
=& P(\frac{X-\mu}{\sigma} < -1) \\
=& P(X < \mu - \sigma)
\end{aligned}
$$

   We know, $P(\mu-\sigma < X < \mu+\sigma) = 0.6827$. So, $P(\mu-\sigma > X > \mu+\sigma) = 1-0.6827 = 0.3173$. By symmetry of normal distribution, $P(X < \mu - \sigma) = P(X > \mu + \sigma)$. So, $P(X < \mu - \sigma) = 0.3173/2 = 0.15865$.

   (b)

$$
\begin{aligned}
P(X > 49.75) =& P(\frac{X-\mu}{\sigma} > \frac{49.75-37}{4.25}) \\
=& P(\frac{X-\mu}{\sigma} > 3) \\
=& P(X > \mu + 3\sigma)
\end{aligned}
$$

   We know, $P(\mu-3\sigma < X < \mu+3\sigma) = 0.9973$. So, $P(\mu-3\sigma > X > \mu+3\sigma) = 1-0.9973 = 0.0027$. By symmetry of normal distribution, $P(X < \mu - 3\sigma) = P(X > \mu + 3\sigma)$. So, $P(X < \mu + 3\sigma) = 0.0027/2 = 0.00135$.

2.

$$
\begin{aligned}
P(32.75 < X < 49.75) =& P(\frac{32.75-37}{4.25} < \frac{X-\mu}{\sigma} < \frac{49.75-37}{4.25}) \\
=& P(-1 < \frac{X-\mu}{\sigma} < 3) \\
=& P(\mu - \sigma < X < \mu + 3\sigma) \\
=& P(\mu - \sigma < X < \mu + \sigma) + P(\mu + \sigma < X < \mu + 3\sigma) \\
=& 0.6827 + P(\mu + \sigma < X < \mu + 3\sigma) \\
P(\mu + \sigma < X < \mu + 3\sigma) =& (P(\mu - 3\sigma < X < \mu + 3\sigma) - P(\mu - \sigma < X < \mu + \sigma))/2 \\
=& (0.9973 - 0.6827)/2 = 0.1573 \\
P(32.75 < X < 49.75) =& 0.6827 + 0.1573 = 0.84.
\end{aligned}
$$

3. The weight of a ream of paper follows a normal distribution with a mean of 2.8 pounds and a standard deviation of 0.8 pounds. Use R to calculate the following questions.

   (a) Approximately what proportion of the reams weigh less than 2.2 pounds?

   (b) What is the probability that a randomly chosen ream will weigh less than 1.234 pounds?

   (c) What is the probability that a randomly chosen ream will weigh between 1.5 and 3.5 pounds?

   (d) If the specification calls for the reams to weigh 2.8 ± 1.6 pounds, use Empirical rule to decide about what proportion of the reams are out-of-specification?

   Solution:

   (a)
   ```
   #P(X < 2.2)
   pnorm(2.2, 2.8, 0.8)
   [1] 0.2266274
   ```

   (b)
   ```
   #P(X < 1.234)
   pnorm(1.234, 2.8, 0.8)
   [1] 0.02514436
   ```

   (c)
   ```
   #P(1.5 < X < 3.5)
   pnorm(3.5, 2.8, 0.8) - pnorm(1.5, 2.8, 0.8)
   [1] 0.7571318
   ```

   (d)

$$
\begin{aligned}
P(2.8 - 1.6 < X < 2.8 + 1.6) =& P(\mu - 2\sigma < X < \mu + 2\sigma) \\
=& 0.9543. \\
P(2.8 - 1.6 > X > 2.8 + 1.6) =& 1 - 0.9543 = 0.0457.
\end{aligned}
$$

4. Suppose X follows normal distribution with a mean of 10 and a standard deviation of 2. Use standardizing method and standard normal table to answer (a), (b), and (c).

   (a) Calculate $P(X < 8.88)$.

   (b) Calculate $P(X < 8.88|X < 12.42)$.

   (c) Find out the value such that 33% of the realizations of X is more than this value.

   (d) Use R to calculate the value in (c).

   Solution:

   (a)

$$
\begin{aligned}
P(X < 8.88) =& P(\frac{X - \mu <}{\sigma} < \frac{8.88 - 10}{2}) \\
=& P(Z < -0.56) = 0.2877
\end{aligned}
$$

(b)

$$\begin{aligned}
P(X < 8.88|X < 12.42) =& \frac{P(X < 8.88 \cap X < 12.42)}{P(X < 12.42)} \\
=& \frac{P(X < 8.88)}{P(X < 12.42)} \\
=& \frac{P(Z < -0.56)}{P(Z < 1.21)} \\
=& \frac{0.2877}{0.8869} \\
=& 0.3243883.
\end{aligned}$$

(c) We need to find c such that:

$$\begin{aligned}
P(X > c) =& 0.33 \\
P(X < c) =& 0.67 \\
P(Z < \frac{c - 10}{2}) =& 0.67 \\
\text{From table } P(Z < 0.44) =& 0.67. \\
\text{Therefore,} \frac{c - 10}{2} =& 0.44 \\
c =& 10.88.
\end{aligned}$$

5. One of the motivation to study normal distribution is the Central Limit Theorem. In this problem, we are going to use R to visualize this important phenomenon.

   (a) Suppose X is a Weibull random variable with shape parameter $\beta = 1.5$ and scale parameter $\delta = 0.5$. We want to have a look of of the shape of the pdf of X. In R, we first use `rweibull(n,`β, δ`)` function to simulate n realizations of X, then use `hist()` function to plot histogram. Let n = 5000. Run the following R code and show the plot. What's the shape of the pdf of X? Is it bell-shaped?

```
x5000 <- rweibull(5000, 1.5, 0.5)
hist(x5000, freq=F, main="Simulated pdf for x")
```

   (b) he Central Limit Theorem says the average value of realizations of X follows normal distribution when the number of realizations is large. In R, we use `mean()` function to calculate the average value of a sequence of numbers. For example,

```
k <- c(1,2,6)
mean(k)
[1] 3
```

   Now, let's generate 1000 realizations of X and find its average:

```
x1000 <- rweibull(1000, 1.5, 0.5)
mean(x1000)
[1] 0.4607919
```

   I claim 0.46 is one observation of the random variable "average value of 1000 realizations of X". Because the values of 1000 X's are random, so the average of them is also random! Define random variable

$$Y = \text{average value of 1000 realizations of X.}$$

Let's see what is the shape of the pdf of Y by simulation. Run the following R code and show the plot. What is the shape of the pdf of Y? It is bell-shaped? Can you see why the Central Limit Theorem is True?

```
y <- rep(0, 5000)
for(i in 1:5000){
x <- rweibull(1000, 1.5, 0.5)
y[i] <- mean(x)
}
hist(y, freq=F, main="Simulated pdf for y")
```

(c) Repeat (a) and (b) again by assuming $X \sim Poisson(1)$. You need to print out two histograms and illustrate the shape of these two histograms. (Hint: the R command to generate 5000 Poisson random variables with parameter 1 is `rpois(5000, 1)`).