Moment Generating Function

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Moments

Moments: We may be interested in some special quantities like $E[X], E[X^2], E[X^3], \ldots, E[X^r]$. These are called the *moments*. $E[X^r]$ is the r^{th} order moment.
For r=1, we get the mean μ. For r=2, $E[X^2] = Var(X) + \mu^2$.

$$E[X^r] = \begin{cases} \sum_x x^r P_X(x); \ discrete \\ \int_{-\infty}^{\infty} x^r f_X(x)dx; \ continuous \end{cases}$$

Instead of calculating everything from scratch, we can use something called the *moment generating function (MGF)*.

Definition (Moment Generating Function)

The MGF of the random variable X is the expected value of e^{tx} and is denoted by $M_X(t)$. That is,

$$M_X(t) = \begin{cases} \sum_x e^{tx} P_X(x); \; \textit{discrete} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x) dx; \; \textit{continuous} \end{cases}$$

Why MGF?

MGF can generate r^{th} order moment (as the name suggests).

$$E[X^r] = \frac{d^r M_X(t)}{dt^r}|_{t=0} = M_X^{(r)}(t)|_{t=0}.$$

That is, if we take r^{th} order derivative of the MGF and set t=0, it will produce r^{th} order moment.

Examples: Binomal

MGF of binomial distribution. Let $X \sim Bin(n, p)$.

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x};\ x = 0, 1, 2, \ldots, n$$

$$M_X(t) = E[e^{tx}] = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

Verify $M_X^{(1)}(t)|_{t=0} = E(X) = np$.

MGF of geometric distribution. Let $X \sim Geom(p)$.

$$P_X(x) = p(1-p)^{x-1};\ x = 1, 2, \dots,$$
$$M_X(t) = E[e^{tx}] = \sum_{x=1}^{\infty} e^{tx} p(1-p)^{x-1}$$

Verify $M_X^{(1)}(t)|_{t=0} = E(X) = 1/p$.

Examples: Exponential

MGF of exponential distribution. Let $X \sim exp(\lambda)$.

$$f_X(x) = \lambda e^{-\lambda x}; \; x \geq 0$$
$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} \lambda e^{-\lambda x} dx$$

Verify $M_X^{(1)}(t)|_{t=0} = E(X) = 1/\lambda$.