Estimating Load-Sharing Properties in a Dynamic Reliability System

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Modeling Dependence Between Components

Most reliability methods are intended for components that operate independently within a system. It is more realistic, however, to develop models that incorporate stochastic dependencies among the system’s components. Options for modeling dependent systems:

- Shock models.
- Load-share models.
Load Sharing Models

- Load share models dictate that component failure rates depend on the operating status of the other system components and the effective system structure function.

- Daniels (1945) originally adopted this model to describe how the strain on yarn fibers increases as individual fibers within a bundle break.

- A bundle of fibers can be considered as a parallel system subject to a steady tensile load.
The Load-Share Rule

The most important element of the load-share model is the rule that governs how failure rates change after some components in the system fail.

- **Equal Load Share Rule**: A constant system load distributed equally among the working components.

- **Local load sharing rule**: A failed component’s load is transferred to adjacent components.

- **Monotone load sharing rule**: The load on any individual component is nondecreasing as other items fail.

Past research has stressed reliability estimation based on known load share rules.
Examples of Load-Share Systems

- **Textile Engineering**: Failure of one back-up system adversely affects another
- **Nuclear Reactor Safety**: Discovery of a major software defect can help reveal or conceal other existing bugs
- **Civil Engineering**: Welded joints on large support structures
- **Materials Testing**: Fatigue and crack growth
- **Population Sampling**: Capture/Recapture methods
- **Combat Modeling**: Loss of component in combat affects death rate of others
An Unknown Load-Share Rule

- Past research emphasizes load-share modeling based on known load share rules.
- In these examples, the load-share rule might be unknown.
- Our focus: Case in which the system is governed by an unknown equal load-share rule.
- General set up: Observe component lifetimes in parallel systems of identical components.
Observe $n$ i.i.d. systems of $k$ components.

For $i = 1, 2, 3, \ldots$, let $S_{i,1} < S_{i,2} < \ldots$ be the successive component failure times for the $i$th system.

$F$ represents the baseline component failure time distribution function.

Hazard function corresponding to $F$ is $R(x) = - \log(1 - F(x))$.

Hazard rate is $r(x) = f(x)/[1 - F(x)]$, where $f(x)$ is the density of $F$. 
Load Share Parameters

Until the first component failure, the failure rate of each of \( k \) components in the system equals the baseline rate \( r(x) \).

Upon the first failure within a system, the failure rates of the \( k-1 \) remaining components jump to \( \gamma_1 r(x) \), and remain at that rate until the next component failure.

After this failure, the failure rates of the \( k-2 \) surviving components jump to \( \gamma_2 r(x) \), and so on.
Load Share Parameters

The (equal) load share rule can be characterized by the \( k - 1 \) unknown parameters \( \gamma = \gamma_1, \gamma_2, \ldots, \gamma_{k-1} \) and the unknown baseline distribution or hazard function.

For example, a system with a constant load would assign

\[
\gamma_j = k/(k-j), \quad j = 1, \ldots, k-1
\]
Maximum Likelihood Estimation of $R$ and $\gamma$

In the $i$th system, the conditional hazard function of the $(j + 1)$ smallest component lifetime $S_{i,j+1}$, given the first $i$ component failure times $S_{i,1}, \ldots, S_{i,j}$, is (for $s > S_{i,j}$)

$$R^* (s | S_{i,1}, \ldots, S_{i,j}) =$$

$$\gamma_j R(s) + (\gamma_{j-1} - \gamma_j) R(S_{i,j}) + \ldots + (1 - \gamma_1) R(S_{i,1})$$

The corresponding likelihood function, in terms of $R^*$, is

$$\prod_{i=1}^{n} \prod_{j=1}^{k} dR^*(S_{ij}) \exp\{ -R^*(S_{ij}) \}.$$
Computing the MLE

Standard approach for finding the MLE:

1. Fix $\gamma$.
2. Maximize likelihood with respect to $R$ to obtain $\hat{R}(\cdot; \gamma)$
3. Plug $\hat{R}(\cdot; \gamma)$ in to obtain the profile likelihood for $\gamma$
4. Compute the MLE $\hat{\gamma}$; final estimator of $R(\cdot)$ is $\hat{R}(\cdot; \hat{\gamma})$

To understand properties of the nonparametric MLE, we model the load-share system using counting processes.
Notation for Counting Processes

- $N_i(t) = \sum_{j=1}^{k} I(S_{i,j} \leq t)$,  \hspace{1em} i = 1, 2, \ldots, n

- $\gamma[N_i(w)] = \sum_{j=0}^{k-1} \gamma_j I(N_i(w) = j)$

- $Y_i(w) = (k - N_i(w-)) I(\tau \geq w)$

- $A_i(t) = \int_0^t \gamma[N_i(u-)] r(u) Y_i(u) du$

If $\gamma$ is known, then analogous to the derivation of the the Nelson-Aalen estimator (with $J(w) = I(\sum_{i=1}^{k} Y_i(w) > 0)$), we obtain the estimator

$$\hat{R}(s; \gamma) = \int_0^s \frac{J(w) dN(w)}{\sum_{i=1}^{n} Y_i(w) \gamma[N_i(w-)]}$$
MLE using Counting Processes

To obtain the estimator of $R(\cdot)$ for the more general case where $\gamma$ is unknown, we first obtain the profile likelihood for $\gamma$ by plugging in $\hat{R}(\cdot; \gamma)$ into the likelihood function.

$$L_p(s; \gamma) = \prod_{i=1}^{n} \prod_{0 \leq w \leq s} \left[ \frac{Y_i(w)\gamma[N_i(w^-)]}{\sum_{l=1}^{n} Y_l(w)\gamma[N_l(w^-)]} \right] dN_i(w).$$

Once $\hat{\gamma}$ is obtained, the estimator of $R$ becomes

$$\hat{R}(s) = \hat{R}(s; \hat{\gamma}).$$
MLE using Counting Processes

By virtue of the product representation of \( \bar{F} = 1 - F \) in terms of \( R \) given by 
\[
\bar{F}(s) = \prod_{0 \leq w \leq s} \left[ 1 - R(dw) \right],
\]
we then obtain an estimator of \( \bar{F} \) via

\[
\hat{F}(s) = \prod_{0 \leq w \leq s} \left[ 1 - \hat{R}(dw) \right].
\]
Solving the MLE

The MLE can be computed by solving the set of $k$ nonlinear equations

\[
U(\tau; \gamma) = \sum_{i=1}^{n} \int_{0}^{\tau} \left[ \frac{Q_i(w)}{\gamma'Q_i(w)} - \frac{Q(w)}{\gamma'Q(w)} \right] dN_i(w) = 0
\]

where

- $Q_{i,j}(t) = Y_i(t)I(N_i(t-) = j)$, $1 \leq i \leq n$, $0 \leq j \leq k - 1$;
- $Q_i(t) = (Q_{i,0}(t), \ldots, Q_{i,k-1}(t))'$, $1 \leq i \leq n$;
- $Q(t) = (\sum_{i=1}^{n} Q_{i,0}(t), \ldots, \sum_{i=1}^{n} Q_{i,k-1}(t))'$;

(Solve with an iterative scheme; e.g. Newton-Raphson.)
Suppose we have that

\[ \hat{\rho}(t; \gamma) = \sum_{i=1}^{n} \gamma * Q_i(t)/\gamma'Q(t); \]

\[ \delta_i(t) = (\delta_{i,0}(t), \ldots, \delta_{i,k-1}(t))', \text{ with} \]
\[ \delta_{i,j}(t) = I(Q_{i,j}(t) > 0), 1 \leq i \leq n; \]

\[ \Upsilon(s; \gamma) \equiv \int_{0}^{s} [D(\rho(w; \gamma)) - \rho(w; \gamma)\rho(w; \gamma)'] \gamma'q(w)dR(w). \]
Lemma:

If \( \{N_i(\cdot), i = 1, \ldots, n\} \) are independent and identically distributed, and

\[
\inf_{0 \leq w \leq \tau} \sum_{j=0}^{k-1} (k - j) \gamma_j P(N_1(w-) = j) > 0,
\]

then \( U(s; \gamma) = \sum_{i=1}^{n} \int_{0}^{s} [\delta_i(w) - \hat{\rho}(w; \gamma)] (dN_i(w) - dA_i(w)) \)

is a square-integrable martingale with quadratic variation process \( \langle U(\cdot; \gamma) \rangle(s) \) whose in-probability limit is \( \Upsilon(s; \gamma) \).

Furthermore, \( n^{-1/2} U(\cdot; \gamma) \) converges weakly to a zero-mean Gaussian process with covariance matrix function \( \Upsilon(\cdot; \gamma) \).
Theorem 1

Under the conditions of the Lemma,

(i) $\hat{\gamma}$ converges in probability to $\gamma$; and

(ii) $\sqrt{n}(\hat{\gamma} - \gamma) \overset{d}{\to} N(0, \Sigma(\tau, \gamma))$ where

$\Sigma(\tau, \gamma) = D(\gamma) \Upsilon(\tau, \gamma)^{-1} D(\gamma)$, and with

$$\Upsilon(\tau, \gamma) \equiv \int_0^\tau \left[ D(\rho(w; \gamma)) - \rho(w; \gamma) \rho(w; \gamma)' \right] \gamma' q(w) dR(w).$$

where

- $\rho(t; \gamma) = E[\sum_{i=1}^{n} \gamma * Q_i(t)] / E[\gamma' Q(t)] = \gamma * q(t)(\gamma/q(t))^{-1}$ and

- $q(s) = (q_0(s), \ldots, q_{k-1}(s))$, with $q_j(w) = E(Q_{i,j}(w)) = (k - j)P(\tau \geq w, N_1(w-) = j)$. 
Theorem 2

Under the conditions of the Lemma, if \( \tau \) is such that \( \gamma'q(\tau) > 0 \), then

\[
\left\{ \sqrt{n}(\hat{R}(s) - R(s)) : 0 \leq s \leq \tau \right\}
\]

converges weakly to a zero-mean Gaussian process with variance function

\[
\Xi(s; \gamma) \equiv \int_0^s \{\gamma'q(w)\}^{-1}dR(w) + \varrho(s; \gamma)'[\Upsilon(\tau; \gamma)]^{-1}\varrho(s; \gamma),
\]

where \( \varrho(s; \gamma) = \int_0^s \rho(w; \gamma)dR(w) \).
Corollary

Under the conditions of Theorem 2,

\[
\left\{ \sqrt{n} (\hat{F}(s) - \bar{F}(s)) : 0 \leq s \leq \tau \right\}
\]

converges weakly to a zero-mean Gaussian process \( \{Z(s) : 0 \leq s \leq \tau \} \) whose variance function is \( \text{Var}\{Z(s)\} = \bar{F}(s)^2 \Xi(s; \gamma) \).
3-Component Parallel System:

2000-2001 Boston Celtics

Paul Pierce
Kenny Anderson
Antoine Walker
Estimated Cumulative Hazard: Minutes Played Until 2nd Foul

Nonparametric Estimate of Hazard Function

Time (t)

Cumulative Hazard (Rhat(t))
Estimated Survivor Function of Minutes Played

Nonparametric Estimate of Survivor Function

Survivor Function ($F_{Bar}(t)$) vs. Time (t)
Confidence Regions (50%, 90%, 95%) for ($\gamma_1$, $\gamma_2$)