IEOR 4500
Maximizing the Sharpe ratio

Suppose we have the setting for a mean-variance portfolio optimization problem:

$$
\begin{aligned}
\mu, & \quad \text{the vector of mean returns} & (1)\\
Q, & \quad \text{the covariance matrix} & (2)\\
\sum_j x_j &= 1, \ \text{(proportions add to 1)} & (3)\\
Ax &\geq b, \ \text{(other linear constraints).} & (4)\\
0 &\leq x. & (5)
\end{aligned}
$$

Note that we can use inequalities (4) to represent, in a generic way, many constraints, including upper bounds on variables (constraints of the form $x_j \leq u_j$), as well as equations and general inequalities of the form $'' \leq''$.

As an alternative to the standard mean-variance problem, we consider a different optimization task. Let r_f be the risk-free interest rate. Consider:

$$
\text{maximize} \ \frac{\mu^T x - r_f}{\sqrt{x^T Q x}} \qquad (6)
$$

s.t.

$$
\begin{aligned}
\sum_j x_j &= 1,\\
Ax &\geq b.\\
0 &\leq x.
\end{aligned}
$$

Problem (6) is difficult because of the nature of its objective. However, under a reasonable assumption, it can be reduced to a standard convex quadratic program.

The **assumption** we make is: there exists a vector x satisfying (3)-(5) such that

$$
\mu^T x - r_f > 0.
$$

This assumption is reasonable: it simply says that our universe of assets is able to beat the risk-free rate of return.

Our approach is as follows: given an asset vector x, define

$$
f(x) = \frac{\mu^T x - r_f}{\sqrt{x^T Q x}}.
$$

Since $\sum_j x_j \ = \ 1$,

$$f(x) = \frac{\mu^T x \ - \ r_f}{\sqrt{x^T Q x}} \ = \ \frac{\mu^T x \ - \ r_f \sum_j x_j}{\sqrt{x^T Q x}} \ = \ \frac{\hat{\mu}^T x}{\sqrt{x^T Q x}},$$

where for each index j, we define $\hat{\mu}_j \ = \ \mu_j \ - \ r_f$.

Using this fact, we note:

Observation: For any vector x with $\sum_j x_j = 1$, and any scalar $\lambda > 0$, $f(\lambda x) \ = \ f(x)$.

To see this, check that if we write $y \ = \ \lambda x$, then $\sqrt{y^T Q y} \ = \ \lambda \sqrt{x^T Q x}$, and similarly $\hat{\mu}^T y = \lambda \hat{\mu}^T x$.

Now we can state our optimization problem. Let $\hat{A}$ be the matrix whose i, j-entry is

$$a_{ij} \ - \ b_i.$$

The problem we consider is:

$$\text{maximize} \ \frac{1}{\sqrt{y^T Q y}} \tag{7}$$

s.t.

$$\hat{\mu}^T y \ = \ 1, \tag{8}$$

$$\hat{A} y \ \geq \ 0. \tag{9}$$

$$0 \ \leq \ y. \tag{10}$$

To see that problems (6) and (7) are indeed equivalent, suppose that $\bar{y}$ is an optimal solution to (7). Notice that because of (8), $\bar{y}$ is not identically zero, and so by (10), $\sum_j \bar{y}_j > 0$. Define the vector

$$\bar{x} \ = \ \frac{\bar{y}}{\sum_j \bar{y}_j}.$$

Then, by construction,

$$\sum_j \bar{x}_j \ = \ 1.$$

Further, since y satisfies (9), then for any row i we have

$$\sum_j (a_{ij} - b_i) \bar{y}_j \ \geq \ 0,$$

or in other words,

$$\sum_j a_{ij} \bar{y}_j \ \geq \ (\sum_j \bar{y}_j) b_i,$$

and as a consequence,

$$\sum_j a_{ij}\bar{x}_j \ge b_i.$$

Therefore, $\bar{x}$ is feasible for problem (6). Further, as we observed before, $f(\bar{x}) = f(\bar{y}) = \frac{1}{\sqrt{y^TQy}}$, since $\hat{\mu}^T\bar{y} = 1$.

In summary: the value of problem (6) is *at least* as large as the value of problem (7). The converse is proved in a similar way. So, indeed, (6) and (7) are equivalent.

So we just have to solve (7). But this is clearly equivalent to:

$$\begin{aligned} & \text{minimize } y^TQy \\ \text{s.t.} \quad & \\ & \hat{\mu}^T y = 1, \\ & \hat{A}y \ge 0. \\ & 0 \le y, \end{aligned}$$

which is just a standard quadratic program.