Recall: If $Y$ is a random variable, then its cumulative distribution function (or c.d.f.) is defined as:

$$F_Y(y) = \text{for any number } y.$$ 

If $Y$ is a continuous r.v., then the probability density function (or p.d.f.) of $Y$ is:

$$f_Y(y) = \text{for any number } y.$$ 

- Often we may be interested in another r.v., say $U$, where $U$ is a function of $Y$, i.e., $U = U(Y)$.
- Or we might be interested in a function of several random variables, e.g., $U(Y_1, Y_2, ..., Y_n)$.

Example: If $Y_1, Y_2, Y_3, Y_4$ are a random sample from a population, we may be interested in the distribution of:

- We will study three methods for determining the distribution of a function of a random variable:

1. The method of c.d.f.'s
2. The method of transformations
3. The method of m.g.f.'s
6.3 The Method of cdf's

- For a function $U$ of a continuous r.v. $Y$ with a known density $f(y)$, we can often find the distribution of $U$ using the definition of the cdf:

Example 1 (Univariate)

$Y =$ amount of sugar produced per day (in tons)

Suppose $f(y) = \begin{cases} 
2y & \text{if } 0 \leq y \leq 1 \\
0 & \text{elsewhere}
\end{cases}$

Let the profit $U = 3Y - 1$ (in hundreds of $\$)

Then $F_U(u) =$

Note $Y$ can range from
Hence $U = 3Y - 1$ can range from

So $F_U(u) =$

and $f_U(u) =$
Example 2: Let $Y$ have the pdf:

$$f(y) = \begin{cases} 6y(1-y), & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the pdf of $U = Y^3$. 
Bivariate situation

- Let \( U = h(Y_1, Y_2) \) be a function of r.v.'s \( Y_1 \) and \( Y_2 \) that have joint density \( f(y_1, y_2) \). We must:
  * Find the values \( (y_1, y_2) \) such that \( U \leq u \).
  * Integrate \( f(y_1, y_2) \) over this region to obtain \( P(U \leq u) = F_u(u) \).
  * Differentiate \( F_u(u) \) to obtain \( f_u(u) \).

Example 1: Let \( Y_1 \) = the amount of gasoline stocked at the beginning of the week. Let \( Y_2 \) = the amount of gasoline sold during the week. The joint density of \( Y_1 \) and \( Y_2 \) is:
\[
f(y_1, y_2) = \begin{cases} 3y_1, & 0 \leq y_2 \leq y_1, \leq 1 \\ 0 & \text{elsewhere} \end{cases}
\]

Find the density of \( U = Y_1 - Y_2 \), the amount of gasoline remaining at the end of the week.

Picture:
- Easier to integrate lower triangular region.
  So $F_u(u) =$

Exercise: Show the expected amount of gasoline remaining at the end of the week is $3/8$. 
Example 2: Suppose the joint density of $Y_1$ and $Y_2$ is:

$$f(y_1, y_2) = \begin{cases} 6e^{-3y_1 - 2y_2} & \text{if } y_1 > 0, y_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the pdf of $U = Y_1 + Y_2$.

Then $F_u(u) =$
Transforming a Uniform \((0,1)\) r.v.

- Recall \(U \sim \text{Unif}(0,1)\) has cdf:
  \[
  F_u(u) = \begin{cases} 
  0, & u < 0 \\
  u, & 0 \leq u \leq 1 \\
  1, & u > 1 
  \end{cases}
  \]

- We can find a transformation \(Y = G(U)\) such that \(G(U)\) has a specified cdf \(F_y(y)\), as long as the inverse \(F^{-1}(y)\) is unique and well-defined.

**Example**: Transform \(U\) into \(Y \sim \text{expon}(\beta)\).

**Note**: \(F_y(y) = \)

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Application: To simulate values \(Y_1, \ldots, Y_n\) from an \(\text{expon}(\beta)\) distribution, just generate \(U_1, \ldots, U_n\) from a \(\text{Unif}(0,1)\) and let \(Y_i = -\beta \ln(1-U_i), \ i=1, \ldots, n\).
Example 3: Let $Y_1$ and $Y_2$ have joint pdf

$$f(y_1, y_2) = \begin{cases} 
5y_1e^{-y_1y_2} & \text{for } 0.2 < y_1 < 0.4, y_2 > 0 \\
0 & \text{elsewhere}
\end{cases}$$

Find the pdf of $U = Y_1Y_2$.

Picture: