Defn: If $Z$ and $W$ are independent, where $Z \sim N(0,1)$ and $W \sim \chi^2_v$, then $T = \frac{Z}{\sqrt{\frac{W}{v}}}$ has a $t$-distribution with $v$ degrees of freedom.

The density of $T$ is:

**Proof:** Use bivariate transformation technique with $U_1 = \frac{Z}{\sqrt{\frac{W}{v}}}$ and auxiliary variable $U_2 = \ldots$

Then:
Exercise: Show that this reduces to the $t$ density.

Lemma: If $W \sim \chi^2_\nu$, then $E\left(\frac{1}{W}\right) = \frac{1}{\nu-2}$ (for $\nu > 2$).

Proof:

Theorem: If $T$ has a $t$-distribution with $\nu$ d.f., then: $E(T) = 0$ (if $\nu > 1$) and $\text{var}(T) = \frac{\nu}{\nu-2}$ (if $\nu > 2$).

Proof:
Note: For $\nu > 2$, $\text{var}(T) = \frac{\nu}{\nu-2} > 1 = \text{var}(Z)$.

- Comparing $t$ density to a standard normal density, both are centered at $\underline{\text{_____}}$ but the $t$-distribution has $\underline{\text{_______}}$ variance.

Picture:

- As $\nu \to \infty$, $\text{var}(T) \to$

- For smaller d.f., the $t$-distribution is $\underline{\text{spread out}}$.

- As $\nu \to \infty$, the $t$-distribution approaches:
The $t$-distribution and the Sample Mean

Theorem: If $Y_1, \ldots, Y_n$ are independent $N(\mu, \sigma^2)$ r.v.'s, then
\[
\frac{\bar{Y} - \mu}{S/\sqrt{n}} \sim t_{n-1}.
\]

Proof:

Example 1(c): Repeat 1(a), but this time assume $s = 26 \text{ ml}$ is the sample standard deviation.
The F-distribution

- Sometimes we have two random samples taken from two normal populations: \( N(\mu_1, \sigma_1^2) \) and \( N(\mu_2, \sigma_2^2) \).

- We may wish to compare the two population variances, \( \sigma_1^2 \) and \( \sigma_2^2 \).

- A reasonable approach is to examine the sample variances, \( s_1^2 \) and \( s_2^2 \).

**Defn:** If \( W_1, W_2 \) are independent r.v.'s, \( W_1 \sim \chi^2_{\nu_1} \) and \( W_2 \sim \chi^2_{\nu_2} \), then \( F = \frac{W_1/\nu_1}{W_2/\nu_2} \) has an F-distribution with \( \nu_1 \) numerator degrees of freedom and \( \nu_2 \) denominator degrees of freedom.

If \( Y \sim F_{\nu_1, \nu_2} \), then its pdf is:

**Proof:** Let \( U_1 = \frac{W_1/\nu_1}{W_2/\nu_2} \), \( U_2 = W_2/\nu_2 \). Use bivariate transformation technique to eventually obtain marginal pdf of \( U_1 \).
Note: If \( \nu_2 > 2 \), then for \( Y \sim F_{\nu_1, \nu_2} \):

\[ E(Y) = \]

- The mean of an F r.v. depends only on the denominator d.f.!
- Also, if \( \nu_2 > 4 \), \( \text{var}(Y) = \]
- Can be shown using \( \text{var}(Y) = E(Y^2) - [E(Y)]^2 \) and properties of \( \chi^2 \) r.v.'s.

Picture of F density:

Properties of F r.v.'s:

1. If \( Y \sim F_{\nu_1, \nu_2} \), then \( \frac{1}{Y} \sim F_{\nu_2, \nu_1} \).

Proof:
② If $Y \sim t_\nu$, then $Y^2 \sim F_{1, \nu}$.

Proof:

③ If $Y \sim F_{\nu_1, \nu_2}$, then
$$\frac{\left(\frac{\nu_1}{\nu_2}\right) Y}{1 + \left(\frac{\nu_1}{\nu_2}\right) Y} \sim \text{Beta}\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right).$$

Theorem: For two independent random samples of sizes $n_1$ and $n_2$ from normal populations with variances $\sigma_1^2$ and $\sigma_2^2$, the statistic

Proof:

Example 3: Two independent samples of sizes 16 and 21 are taken from two normal populations with $\sigma_1^2 = 3$ and $\sigma_2^2 = 5$. What is the probability that $S_2^2$ is at least twice as large as $S_1^2$?