Chapter 8: Estimation

Goal of Statistical Inference: Use information in our sample data to make a conclusion about a population of interest.

- In particular, once we have chosen a distribution to be a model for our data, we want to estimate any unknown parameter(s) of that distribution.

Example 1: A company wishes to estimate the mean service time \( \mu \) for customers.

- The mean, \( \mu \), is the _______ ________ (or parameter of interest).

Example 2: A manufacturer wishes to estimate the standard deviation \( \sigma \) of the diameters of a part produced in a factory.

- Here, \( \sigma \) is the _______ ________ .

Two Types of Estimate

- A point estimate is a single number.

- An interval estimate is a range of possible values, such as:
- An estimator is a formula for calculating an estimate from data values in a sample.
- The estimate is the actual calculated value.

**Judging the Quality of Point Estimators**

- We traditionally evaluate estimators based on their values across repeated samples (of the same size) from the same population.
- Since an estimator is a function of the random sample values, it is itself a random variable.
- We want a point estimator to be ______ — to be at or near the target parameter value, on average.
- We also want our point estimator to be ______ — to be consistent in value across repeated samples.
- Good precision $\equiv$ _____ variance.
8.2 Bias and Mean Squared Error of a Point Estimator

- Suppose our target parameter is denoted \( \theta \).
- Then let a point estimator of \( \theta \) be denoted by \( \hat{\theta} \).
- Since \( \hat{\theta} \) is a r.u., we can find \( E(\hat{\theta}) \) and \( \text{var}(\hat{\theta}) \).
- We say a point estimator \( \hat{\theta} \) is **unbiased** if:

  (if the average value of the estimator equals the target parameter)

- The **bias** of an estimator \( \hat{\theta} \) is:

- While being unbiased is good, unbiasedness alone does not make an estimator desirable.
- An estimator that is far less than \( \theta \) half the time and equally far above \( \theta \) the other half of the time is unbiased, but it is not a good estimator of \( \theta \).
- We also want our estimator to have low variance.
The Mean Squared Error (MSE) of an estimator measures a combination of bias and variance:

Note:

So the MSE of \( \hat{\theta} \) equals its variance plus the square of its bias.
Example 1: Let $Y$ be a single observation from a binomial distribution with known $n$ and unknown $p$. (That is, out of $n$ "trials", $Y$ is our observed number of "successes".) We wish to estimate the true success probability $p$.

- Let $\hat{p} = \frac{Y}{n}$. Then

- Another estimator of $p$ could be $\hat{p}^* = \frac{Y+1}{n+2}$.

Note:

But note:

Exercise: For a given value of $n$, plot $\text{MSE}(\hat{p}) - \text{MSE}(\hat{p}^*)$ against $p$. See that for values of $p$ near 0.5, then $\hat{p}^*$ has lower MSE than $\hat{p}$. 
8.3 Some Common Unbiased Estimators

Situation 1: \( Y_1, \ldots, Y_n \) iid with mean \( \mu \) and variance \( \sigma^2 \).

Then

Situation 2: If \( X_1, \ldots, X_n \) iid Bernoulli \((p)\) and \( Y = \sum_{i=1}^{n} X_i \),

then:

Situation 3:

Two indep. samples: \( Y_{11}, \ldots, Y_{1n_1} \) iid, mean \( \mu_1 \) and variance \( \sigma_1^2 \)

and \( Y_{21}, \ldots, Y_{2n_2} \) iid, mean \( \mu_2 \) and variance \( \sigma_2^2 \)

Then:

Note:

Situation 4:

Two indep. samples: \( X_{11}, \ldots, X_{1n_1} \) iid Bernoulli \((p_1)\),

let \( Y_1 = \sum_{i} X_{1i} \)

and \( X_{21}, \ldots, X_{2n_2} \) iid Bernoulli \((p_2)\),

let \( Y_2 = \sum_{i} X_{2i} \)

Then:

Note:
Situation 1 again: The sample variance
\[ s^2 = \frac{\sum_{i=1}^{n} (Y_i - \bar{Y})^2}{n-1} \]
is an unbiased estimator of \( \sigma^2 \).
Proof:
Why not use \[ \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2 \]
to estimate \( \sigma^2 \)?

Note:

Interesting Fact: Although the sample variance \( S^2 \) is an unbiased estimator of \( \sigma^2 \), the sample standard deviation \( S = \sqrt{S^2} \) is a biased estimator of the population standard deviation \( \sigma \).