Let $E = |\hat{\theta} - \theta|$ be the error in estimation for an estimator $\hat{\theta}$ of $\theta$.

**Empirical Rule**: Suppose $\hat{\theta}$ has an approximately normal sampling distribution with mean $\theta$ and variance $\sigma^2_{\hat{\theta}}$.

Then:

- These follow from basic normal probabilities.

**Example 1**: To estimate an unknown population mean $\mu$, we take a random sample of size 49, and we calculate $\overline{Y}$. What is an approximate 95% bound on $E$?
Note: If $\hat{\theta}$ does not have an approximately normal sampling distribution, we can still get a conservative bound on $\mathbb{E}$.

Chebyshev's Inequality says:

for $k > 0$.

So if $k=2$, then
8.5  **Confidence Intervals**

- A **confidence interval** (or **interval estimator**) is an interval of numbers containing "reasonable values" for an unknown parameter (which we may generally label $\theta$).

- We would like our interval estimator to:
  1. have a high probability of containing the true value of $\theta$
  2. be relatively narrow (precise)

- A random interval $[\hat{\theta}_L, \hat{\theta}_U]$ is a $100(1-\alpha)\%$ CI for $\theta$ if:

  - The number $1-\alpha$ is called the **confidence coefficient**.
  - We often choose $1-\alpha$ to be large, like:

- We could also have **one-sided** CIs like $[\hat{\theta}_L, \infty)$ or $(-\infty, \hat{\theta}_U]$ where

  \[ P(\hat{\theta}_L \leq \theta) = 1-\alpha \]  or  \[ P(\theta \leq \hat{\theta}_U) = 1-\alpha. \]
Pivotal Method

- A useful method for deriving confidence intervals is to use a **pivotal quantity**:

- A pivotal quantity
  1. is a function of the sample data, the unknown target parameter, and **no other** unknown quantities.
  2. has a distribution that does not depend on the target parameter.

Example 1: We will randomly sample \( n = 1 \) observation from an exponential distribution with unknown mean \( \theta \). Find a formula for a 90% CI for \( \theta \).

If \( Y \sim \text{expon}(\theta) \), then \( f_Y(y) = \)

- It is easily shown that the density of \( U = \frac{Y}{\theta} \) is:

- Note \( U = \frac{Y}{\theta} \) is clearly a pivotal quantity.
- We need to find two numbers a and b such that:

Picture:

- One idea: Set

Then:

Solve for a and b:

So:

- We want to isolate θ in the middle:
Example 2: We sample n=1 observation from a $\text{Unif}(0, \Theta)$ distribution where $\Theta$ is unknown. Find a 95% lower confidence bound for $\Theta$. $Y \sim \text{Unif}(0, \Theta)$. It can be shown that 
\[
\frac{Y}{\Theta} \sim
\]

Hence $\frac{Y}{\Theta}$ is a pivotal quantity.

- We want some number $a$ such that:

- If we observe $Y = 3.8$ from this uniform distribution, then our 95% lower confidence bound is:

- Hence we are 95% confident that $\Theta$ is
Example 3: Let $X_1, \ldots, X_{10}$ be iid r.v.'s from an exponential distribution with unknown mean $\theta$. Use the pivotal method to find a 95% CI for $\theta$.
- Recall from Sec. 6.5: If $X_1, \ldots, X_n \text{iid expon}(\theta)$, then:

- Also from Sec. 6.5, if
Example 4: Let $Y_1, \ldots, Y_n$ be iid Unif $(0, \theta)$. Use the pivotal method to find a 95% upper confidence bound for $\theta$.

- Recall the maximum, $Y_{(n)}$, has cdf:

- Hence consider $U = F_u(u) = \ldots$

Example: If we have $n=10$, with the maximum $Y_{(n)} = 5.7$, we are 95% confident that $\theta$ is at most: