9.5 The Rao-Blackwell Theorem and MVUEs

This theorem allows us to find unbiased estimators with small variances.

**Rao-Blackwell Theorem:** Let \( \hat{\theta} \) be an unbiased estimator of \( \theta \) with \( \text{var}(\hat{\theta}) < \infty \). Let \( U \) be a sufficient statistic for \( \theta \). Then define \( \hat{\theta}^* = E(\hat{\theta} | U) \). Then:

**Proof:**
The Rao-Blackwell Theorem says that if we have an unbiased estimator, we can get an unbiased estimator whose variance is at least as small by conditioning on a sufficient statistic.

Which choice of sufficient statistic should we use?

A minimal sufficient statistic is the sufficient statistic that "condenses the data" more than any other sufficient statistic.

Definition: A sufficient statistic is minimal sufficient if it is a function of every other sufficient statistic.

Definition (Completeness): A statistic \( U \) is complete if and only if

Note: Any complete sufficient statistic is minimal sufficient.
- The Lehmann-Scheffé Theorem says that if we can construct an unbiased estimator that is a function of a ____________, then this will be the Minimum Variance Unbiased Estimator (MVUE) of θ.

- Useful Result: If Y₁, ..., Yₙ is a random sample from a distribution in the one-parameter exponential family, i.e., having pdf of the form:

then:

- This result is useful for many examples!
Proof: (We prove the sufficiency of $\sum_{i=1}^{n} d(y_i)$; the proof of completeness is quite advanced.)

Example 1: $Y_1, \ldots, Y_n \overset{iid}{\sim} \text{Pois}(\lambda)$. Find the MVUE for $\lambda$. 
Example 2: \( Y_1, \ldots, Y_n \overset{iid}{\sim} \text{Weibull}(\theta) \), with \( m = 2 \) (see exercise 6.26). Find the MVUE for \( \theta \).
The Lehmann-Scheffé theorem can also be used to find MVUEs for functions of parameters of some distribution.

Example 3: $Y_1, \ldots, Y_n \overset{iid}{\sim} \text{Expon} (\beta)$. Find the MVUE for $\beta^2 \ (= \text{var} (Y_i))$.
Example 4: \( Y_1, \ldots, Y_n \sim N(\mu, 1) \). Find the MVUE for \( \mu^2 \).
Example 5: Let $Y_1, \ldots, Y_n \iid \text{Unif } (0, \theta)$. Find a complete sufficient statistic for $\theta$, and then find the MVUE of $\theta$.

Note: The $\text{Unif}(0, \theta)$ distribution is not in the exponential family (its support ). But we can show $U = Y_{(n)} = \max \{Y_1, \ldots, Y_n\}$ is sufficient via the definition of sufficiency. We must show:
We prove $U = Y_n$ is complete using the definition of completeness:
9.6 The Method of Moments

- The method of moments is a simple way to obtain point estimators.
- It involves equating population moments to sample moments and solving the equations.
- If we have to estimate $t$ parameters, we set up $t$ equations:

- Then solve these equations simultaneously for the desired parameters.
- Example 1: $Y_1, ..., Y_n \overset{iid}{\sim} \text{Unif}(0, \theta)$. Find the MME of $\theta$. 
Example 2: \( Y_1, \ldots, Y_n \ \text{iid} \ \Gamma(\alpha, \beta) \). Find MMEs of \( \alpha \) and \( \beta \).
Example 3: Let $Y_1, \ldots, Y_n$ be iid r.v.'s with pdf
\[ f_Y(y) = \begin{cases} \frac{2}{\theta^2}(\theta - y), & 0 < y < \theta \\ 0 & \text{elsewhere} \end{cases} \]
Find the MME of $\theta$.

- Typically MMEs have low bias for large sample sizes.
- But... often MMEs are not functions of sufficient statistics, so they have poor efficiency relative to other possible estimators.