1. Define the matrix
   \[ A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \]

   (a) Find two generalized inverses of \( A \).

   (b) Find a matrix which projects onto \( C(A) \).

   (c) Find a matrix which projects onto \( C(A)^\perp \), the orthogonal complement of \( C(A) \).

2. Show that if \( A^- \) is a generalized inverse of \( A \), then so is
   \[ G = A^- AA^- + (I - A^- A)B_1 + B_2(I - AA^-), \]
   for any choices of \( B_1 \) and \( B_2 \) with conformable dimensions.

3. Let \( A_{n \times p}, b_{p \times 1}, c_{n \times 1} \), and suppose that the equations \( Ab = c \) are consistent. Let \( x_{n \times 1}, u_{p \times 1}, \) and \( X_{p \times n} \). Let \( A_1^- \) and \( A_2^- \) be two generalized inverses of \( A \). Let \( I \) denote the \( n \times n \) identity matrix.

   (a) Let \( b^* \) be a solution to \( Ab = c \). Show that \( b^* + uc'(A_1^-)'A' - I)x \) is also a solution.

   (b) Show that \( A_1^- + X( AA_2^- - I) \) is a generalized inverse of \( A \).

4. Suppose the system \( Ax = c \) is consistent and that \( G \) is a generalized inverse of \( A \).

   (a) What is a particular solution to the system? the general solution?

   (b) If \( A \) is symmetric, prove that \( \frac{1}{2}(G + G') \) is a generalized inverse of \( A \).

   (c) Prove that the generalized inverse in (b) is symmetric. This shows that there does exist a generalized inverse of \( A \), \( A \) symmetric, that is symmetric itself.

5. Suppose that \( A, B, \) and \( A + B \) are all idempotent. Prove that \( AB = 0 \) and \( BA = 0 \).

6. Let \( P \) be an \( n \times n \) orthogonal matrix and let \( A \) be an \( n \times n \) symmetric and idempotent matrix. Define \( D = P'AP \). Show that \( D \) is a perpendicular projection matrix.

7. Consider the linear model \( Y = X\beta + \epsilon \) with
   \[ Y = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \]

   Note that \( r(X) = 3 \). Find \( \hat{\beta}_1 \) and \( \hat{\beta}_2 \), two different solutions to the normal equations \( X'X\beta = X'Y \). With your solutions, show that \( X\hat{\beta}_1 = X\hat{\beta}_2 \in C(X) \). Also show that \( Y - X\hat{\beta}_1 = Y - X\hat{\beta}_2 \in N(X') \).
8. Let $M_1$ and $M_2$ be perpendicular projection matrices on $\mathcal{R}^n$. Prove that $M_1 + M_2$ is the perpendicular projection matrix onto $\mathcal{C}(M_1, M_2)$ if and only if $\mathcal{C}(M_1) \perp \mathcal{C}(M_2)$.

9. Let $M$ be the perpendicular projection matrix onto $\mathcal{C}(X)$. Suppose that $a \in \mathcal{C}(X)$. Show that $(M - aa')'(M - aa') = M + (a'a - 2)aa'$.

10. Suppose that $M_1$ and $M_2$ are symmetric, that $\mathcal{C}(M_1) \perp \mathcal{C}(M_2)$, and that $M_1 + M_2$ is the perpendicular projection matrix. Prove that $M_1$ and $M_2$ are also perpendicular projection matrices.

11. Let $M$ and $M_0$ be perpendicular projection matrices with $\mathcal{C}(M_0) \subset \mathcal{C}(M)$. Show that $M - M_0$ is a perpendicular projection matrix.

**DEFINITION:** Let $\mathcal{V}$ denote an arbitrary vector space and let $\mathcal{S}$ denote a subspace of $\mathcal{V}$. Define

$$\mathcal{S}_\perp = \{ y \in \mathcal{V} : y \perp \mathcal{S} \}.$$  

The subspace $\mathcal{S}_\perp$ is called the **orthogonal complement of $\mathcal{S}$ with respect to $\mathcal{V}$**. If $\mathcal{V} = \mathcal{R}^n$, then $\mathcal{S}_\perp = \mathcal{S}^\perp$; in this situation, we call $\mathcal{S}^\perp$ and $\mathcal{S}$ simply “orthogonal complements” because it is understood that the larger vector space is $\mathcal{R}^n$. However, there is nothing to prevent $\mathcal{V}$ from being a subspace of $\mathcal{R}^n$.

12. Let $M$ and $M_0$ be perpendicular projection matrices with $\mathcal{C}(M_0) \subset \mathcal{C}(M)$. Show that $\mathcal{C}(M - M_0) = \mathcal{C}(M_0)^\perp_{\mathcal{C}(M)}$, the orthogonal complement of $\mathcal{C}(M_0)$ with respect to $\mathcal{C}(M)$.