1. Suppose that $A$ is an $n \times n$ symmetric matrix. Prove that $A$ is idempotent if and only if $r(A) + r(I - A) = n$.

2. Let $P$ and $A$ be $n \times n$ matrices. Define $D = P'AP$. Show that if $A$ is nnd, then so is $D$.

3. Consider the matrix

$$A = \begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & -1 \\
-1 & -1 & 3
\end{pmatrix}.$$ 

(a) Show that $A$ is pd.
(b) Compute $A^{1/2}$, the symmetric square root of $A$. Check your work by showing that $A^{1/2}A^{1/2} = A$.

4. Suppose that $A_{n \times n}$ is a symmetric with eigenvalues $\lambda_{(1)} < \lambda_{(2)} < \cdots < \lambda_{(n)}$. Prove that

$$\sup_{x \neq 0} \frac{x'Ax}{x'x} = \lambda_{(n)}.$$

5. Prove that if a matrix $A$ is pd, then $A^{-1}$ is also pd.

6. Define

$$A = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
1 & 1 & 0
\end{pmatrix}.$$ 

Suppose that $M$ is the perpendicular projection matrix onto $C(A)$. Find $r(M)$ and $tr(M)$.

7. Suppose that $Y = (Y_1, Y_2, Y_3)'$ has mean $\mu$ and covariance matrix $\Sigma$ given by

$$\mu = \begin{pmatrix}
4 \\
6 \\
10
\end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix}
8 & 5 & 0 \\
5 & 12 & 4 \\
0 & 4 & 9
\end{pmatrix}.$$ 

(a) Find the mean and variance of $Z = Y_1 - Y_2 + Y_3$.
(b) Let

$$A = \begin{pmatrix}
3 & 5 & 4 \\
1 & 2 & 8
\end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix}
-1 \\
2
\end{pmatrix}.$$ 

Find $E(AY + b)$ and $cov(AY + b)$.

8. Suppose that $Y = (Y_1, Y_2, ..., Y_n)'$ is a random vector with covariance matrix $\Sigma = \text{cov}(Y)$, and let $a$ and $b$ be conformable vectors of constants. Prove that

$$\text{cov}(a'Y, b'Y) = a'\Sigma b.$$
9. Suppose that $Y_{n \times 1}$ and $X_{k \times 1}$ are random vectors. Define $Z = Y - E(Y|X)$. Show that $Z$ and $X$ are uncorrelated.

10. Suppose that $Y$ and $X$ are random vectors with means $\mu_Y$ and $\mu_X$, respectively, variance matrices $\Sigma_Y$ and $\Sigma_X$, respectively, and covariance matrix $\Sigma_{YX}$. Assume that $\Sigma_X$ is nonsingular. Define

$$W = \mu_Y + \Sigma_{YX} \Sigma_X^{-1} (X - \mu_X)$$

and $Z = Y - W$. Derive $\text{cov}(Z)$ and show that $\text{cov}(Z) \preceq_{pd} \Sigma_Y$, with equality when $\Sigma_{YX} = 0$.

11. Consider the mixed-effects linear model

$$Y_{n \times 1} = X\beta + Z_1 \epsilon_1 + \epsilon_2,$$

where $X$ is $n \times p$, $\beta$ is $p \times 1$, $\epsilon_1$ has mean vector $0_{r \times 1}$ and variance-covariance matrix $\Sigma_1$, and $\epsilon_2$ has mean vector $0_{n \times 1}$ and variance-covariance matrix $\sigma^2 I_n$. Also, assume that $\epsilon_1$ and $\epsilon_2$ are uncorrelated.

(a) Compute $\text{cov}(Y)$.
(b) (†) Specialize to the one-factor random-effects model

$$Y_{ij} = \mu + \alpha_i + \epsilon_{ij},$$

for $i = 1, 2, 3$ and $j = 1, 2$, where $\alpha_1, \alpha_2, \alpha_3$ are iid $\mathcal{N}(0, \sigma_\alpha^2)$, $\epsilon_{ij}$ are iid $\mathcal{N}(0, \sigma^2)$, and the $\alpha_i$'s and $\epsilon_{ij}$'s are mutually independent. Put this model into the form $Y_{n \times 1} = X\beta + Z_1 \epsilon_1 + \epsilon_2$, and compute $\text{cov}(Y)$. 