CHAPTER 2

2.2 Sample spaces

**TERMINOLOGY:** Suppose that a random experiment is performed and that we observe an outcome from the experiment (e.g., rolling a die). The set of all possible outcomes for an experiment is called the sample space and is denoted by $S$.

**Example 2.2.** In each of the following random experiments, we write out a corresponding sample space.

(a) The Michigan state lottery calls for a three-digit integer to be selected:

$$S = \{000, 001, 002, \ldots, 998, 999\}.$$ 

(b) A USC student is tested for chlamydia (0 = negative, 1 = positive):

$$S = \{0, 1\}.$$ 

(c) An industrial experiment consists of observing the lifetime of a battery, measured in hours. Different sample spaces are:

$$S_1 = \{w : w \geq 0\} \quad S_2 = \{0, 1, 2, 3, \ldots\} \quad S_3 = \{\text{defective, not defective}\}.$$ 

Sample spaces are not unique; in fact, how we describe the sample space has a direct influence on how we assign probabilities to outcomes in this space. $\Box$

2.3 Basic set theory

**TERMINOLOGY:** A countable set $A$ is a set whose elements can be put into a one-to-one correspondence with $\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of natural numbers. A set that is not countable is said to be uncountable.

**TERMINOLOGY:** Countable sets can be further divided up into two types.

- A **countably infinite set** has an infinite number of elements.
- A **countably finite set** has a finite number of elements.
Example 2.3. Say whether the following sets are countable (and, furthermore, finite or infinite) or uncountable.

(a) \( A = \{0, 1, 2, \ldots, 10\} \) **countable**.

(b) \( B = \{1, 2, 3, \ldots\} \) **countably infinite**.

(c) \( C = \{x : 0 < x < 2\} \) **interval** uncountable.

**TERMINOLOGY:** Suppose that \( A \) and \( B \) are sets (events). We say that \( A \) is a **subset** of \( B \) if every outcome in \( A \) is also in \( B \), written \( A \subseteq B \). (\( A \subseteq B \))

- **Implication:** In a random experiment, if the event \( A \) occurs, then so does \( B \). The converse is not necessarily true.

**TERMINOLOGY:** The **null set** denoted by \( \emptyset \) is the set that contains no elements.

**TERMINOLOGY:** The **union** of two sets \( A \) and \( B \) is the set of all elements in either \( A \) or \( B \) (or both), written \( A \cup B \). The **intersection** of two sets \( A \) and \( B \) is the set of all elements in both \( A \) and \( B \), written \( A \cap B \). Note that \( A \cap B \subseteq A \cup B \).

- **Remember:** Union \( \rightarrow \) "or"  Intersection \( \rightarrow \) "and"

**EXTENSION:** We extend the notion of unions and intersections to more than two sets.

Suppose that \( A_1, A_2, \ldots, A_n \) is a **finite** sequence of sets. The union of \( A_1, A_2, \ldots, A_n \) is

\[
\bigcup_{j=1}^{n} A_j = A_1 \cup A_2 \cup \cdots \cup A_n,
\]

that is, the set of all elements contained in at least one \( A_j \). The intersection of \( A_1, A_2, \ldots, A_n \) is

\[
\bigcap_{j=1}^{n} A_j = A_1 \cap A_2 \cap \cdots \cap A_n,
\]

the set of all elements contained in each of the sets \( A_j, j = 1, 2, \ldots, n \).
**EXTENSION:** Suppose that \( A_1, A_2, \ldots \) is a countable sequence of sets. The union and intersection of this infinite collection of sets is denoted by

\[
\bigcup_{j=1}^{\infty} A_j \quad \text{and} \quad \bigcap_{j=1}^{\infty} A_j,
\]

respectively. The interpretation is the same as before.

**Example 2.4.** Define the sequence of sets \( A_j = [1 - 1/j, 1 + 1/j] \), for \( j = 1, 2, \ldots \). Then,

\[
\bigcup_{j=1}^{\infty} A_j = [0, 2] \quad \text{and} \quad \bigcap_{j=1}^{\infty} A_j = \{1\}. \quad \square
\]

**TERMINOLOGY:** Suppose that \( A \) is a subset of \( S \) (the sample space). The complement of \( A \) is the set of all elements not in \( A \) (but still in \( S \)). We denote the complement by \( \overline{A} \).

**Distributive Laws:**

1. \( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \)
2. \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \)

**DeMorgan’s Laws:**

1. \( \overline{A \cap B} = \overline{A} \cup \overline{B} \)
2. \( \overline{A \cup B} = \overline{A} \cap \overline{B} \)

**TERMINOLOGY:** We call two events \( A \) and \( B \) mutually exclusive, or disjoint, if \( A \cap B = \emptyset \), that is, \( A \) and \( B \) have no common elements.

**Example 2.5.** Suppose that a fair die is rolled. A sample space for this random experiment is \( S = \{1, 2, 3, 4, 5, 6\} \).

(a) If \( A = \{1, 2, 3\} \), then \( \overline{A} = \{4, 5, 6\} \).

(b) If \( A = \{1, 2, 3\} \), \( B = \{4, 5\} \), and \( C = \{2, 3, 6\} \), then \( A \cap B = \emptyset \) and \( B \cap C = \emptyset \). Note also that \( A \cap B \cap C = \emptyset \) and \( A \cup B \cup C = S \). \quad \square

Moreover, we have:

\[
\bigcap_{j=1}^{\infty} A_j = \{1\}.
\]

Obviously, \( 1 \) is in \( \bigcap_{j=1}^{\infty} A_j \).

If \( x \notin A_1 \), \( x \notin A_j \) for all \( j \).

Then, for all \( j \),

\[
x \text{ is in } A_j = \left[1 - \frac{1}{j}, 1 + \frac{1}{j}\right] \quad \text{such that } \frac{1}{j} < |x - 1|.
\]

Take \( j \) large enough such that \( \frac{1}{j} < |x - 1| \), then \( x > 1 + \frac{1}{j} \) or \( x < 1 - \frac{1}{j} \).

i.e. \( x \) is not in \( A_j \)

**Contradiction:**

So \( \bigcap_{j=1}^{\infty} A_j \) can only be 1.