CHAPTER 2  STAT/MATH 511, J. TEBBS

SOLUTION: First, the number of sample points in $S$ is given by
\[ N = \binom{20}{5} = \frac{20!}{5! (20 - 5)!} = 15504. \]

Let $A$ denote the event that the lot is accepted. How many ways can $A$ occur? Use the multiplication rule.

Stage 1 Choose 5 good drives from 17 \[ \binom{17}{5} \]
Stage 2 Choose 0 bad drives from 3 \[ \binom{3}{0} \]

By the multiplication rule, there are \[ n_a = \binom{17}{5} \times \binom{3}{0} = 6188 \] different ways $A$ can occur. Assuming an equiprobability model (i.e., each outcome is equally likely), $P(A) = n_a/N = 6188/15504 \approx 0.399$. \(\square\)

2.7 Conditional probability

MOTIVATION: In some problems, we may be fortunate enough to have prior knowledge about the likelihood of events related to the event of interest. We may want to incorporate this information into a probability calculation.

TERMINOLOGY: Let $A$ and $B$ be events in a nonempty sample space $S$. The conditional probability of $A$, given that $B$ has occurred, is given by
\[ P(A|B) = \frac{P(A \cap B)}{P(B)}, \]
provided that $P(B) > 0$.

Example 2.21. A couple has two children.
Example 2.21. A couple has two children.

(a) What is the probability that both are girls? \[ \frac{1}{4} \]

(b) What is the probability that both are girls, if the eldest is a girl?

---

CHAPTER 2

SOLUTION: (a) The sample space is given by

\[ S = \{(M,M), (M,F), (F,M), (F,F)\} \]

and \( N = 4 \), the number of sample points in \( S \). Define

\[ A_1 = \{ \text{1st born child is a girl}\}, \]
\[ A_2 = \{ \text{2nd born child is a girl}\}. \]

Clearly, \( A_1 \cap A_2 = \{(F,F)\} \) and \( P(A_1 \cap A_2) = 1/4 \), assuming that the four outcomes in \( S \) are equally likely.

SOLUTION: (b) Now, we want \( P(A_2|A_1) \). Applying the definition of conditional probability, we get

\[ P(A_2|A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)} = \frac{1/4}{2/4} = 1/2. \]

Example 2.22. In a certain community, 36 percent of the families own a dog, 22 percent of the families that own a dog also own a cat, and 30 percent of the families own a cat. A family is selected at random.

(a) Compute the probability that the family owns both a cat and dog.

(b) Compute the probability that the family owns a dog, given that it owns a cat.

SOLUTION: Let \( C = \{ \text{family owns a cat}\} \) and \( D = \{ \text{family owns a dog}\} \). From the
SOLUTION: Let $C = \{\text{family owns a cat}\}$ and $D = \{\text{family owns a dog}\}$. From the problem, we are given that $P(D) = 0.36$, $P(C|D) = 0.22$ and $P(C) = 0.30$. In (a), we want $P(C \cap D)$. We have
\[
0.22 = P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{P(C \cap D)}{0.36}.
\]
Thus,
\[
P(C \cap D) = 0.36 \times 0.22 = 0.0792.
\]
For (b), we want $P(D|C)$. Simply use the definition of conditional probability:
\[
P(D|C) = \frac{P(C \cap D)}{P(C)} = \frac{0.0792}{0.30} = 0.264. \square
\]

CHAPTER 2

RESULTS: It is interesting to note that conditional probability $P(\cdot|B)$ satisfies the axioms for a probability set function when $P(B) > 0$. In particular,

1. $P(A|B) \geq 0$
2. $P(B|B) = 1$
3. If $A_1, A_2, \ldots$ is a countable sequence of pairwise mutually exclusive events (i.e., $A_i \cap A_j = \emptyset$, for $i \neq j$) in $S$, then
   \[
   P\left(\bigcup_{i=1}^{\infty} A_i \bigg| B\right) = \sum_{i=1}^{\infty} P(A_i|B).
   \]

EXERCISE. Show that the measure $P(\cdot|B)$ satisfies the Kolmogorov axioms when $P(B) > 0$; i.e., establish the results above.

MULTIPLICATION LAW OF PROBABILITY: Suppose $A$ and $B$ are events in a non-empty sample space $S$. Then,
\[
P(A \cap B) = P(B|A)P(A) = P(A|B)P(B).
\]
\[ P(A \cap B) = P(B|A)P(A) \]
\[ = P(A|B)P(B). \]

Proof. As long as \( P(A) \) and \( P(B) \) are strictly positive, this follows directly from the definition of conditional probability. \( \Box \)

**EXTENSION**: The multiplication law of probability can be extended to more than 2 events. For example,

\[
P(A_1 \cap A_2 \cap A_3) = P[(A_1 \cap A_2) \cap A_3] = P(A_3|A_1 \cap A_2) \times P(A_1 \cap A_2) = P(A_3|A_1 \cap A_2) \times P(A_2|A_1) \times P(A_1).
\]

**NOTE**: This suggests that we can compute probabilities like \( P(A_1 \cap A_2 \cap A_3) \) “sequentially” by first computing \( P(A_1) \), then \( P(A_2|A_1) \), then \( P(A_3|A_1 \cap A_2) \). The probability of a \( k \)-fold intersection can be computed similarly; i.e.,

\[
P \left( \bigcap_{i=1}^{k} A_i \right) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \times \cdots \times P \left( A_k \bigg| \bigcap_{i=1}^{k-1} A_i \right).
\]

---

**CHAPTER 2**

**Example 2.23.** I am dealt a hand of 5 cards. What is the probability that they are all spades?

**SOLUTION.** Define \( A_i \) to be the event that card \( i \) is a spade \((i = 1, 2, 3, 4, 5)\). Then,

\[
P(A_1) = \frac{13}{52}, \quad P(A_2|A_1) = \frac{12}{51}, \quad P(A_3|A_1 \cap A_2) = \frac{11}{50}, \quad P(A_4|A_1 \cap A_2 \cap A_3) = \frac{10}{49}, \quad P(A_5|A_1 \cap A_2 \cap A_3 \cap A_4) = \frac{9}{48},
\]

so that
so that

$$P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) = \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} \times \frac{9}{48} \approx 0.0005.$$  

**Note:** As another way to solve this problem, a student recently pointed out that we could simply regard the cards as belonging to two groups: spades and non-spades. There are \( \binom{13}{5} \) ways to draw 5 spades from 13. There are \( \binom{52}{5} \) possible hands. Thus, the probability of drawing 5 spades (assuming that each hand is equally likely) is \( \frac{\binom{13}{5}}{\binom{52}{5}} \approx 0.0005. \square \)

### 2.8 Independence

**Terminology:** When the occurrence or non-occurrence of \( A \) has no effect on whether or not \( B \) occurs, and vice versa, we say that the events \( A \) and \( B \) are **independent**. Mathematically, we define \( A \) and \( B \) to be independent iff

$$P(A \cap B) = P(A)P(B).$$

Otherwise, \( A \) and \( B \) are called **dependent** events. Note that if \( A \) and \( B \) are independent,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

and

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)P(A)}{P(A)} = P(B).$$