Exam 1 (Due April 08, 2015, 10:40 am) 
Name:

Rules: 1. You have exactly 14 days to solve the exam. It has to be received by me before or at the end of the class on April 8th, 2015. Late submission will be severely penalized. 2. The exam must be taken completely alone. Showing it or discussing it with anybody (except the instructor) is forbidden. 3. Make an effort to make your solution clear and readable. Severe readability issues may be penalized by grade.

1. Let \( \{W_t\} \) be a stationary zero-mean time series. Define two processes

   \[
   X_t = W_t - aW_{t-1} - bW_{t-2}
   \]

   and

   \[
   Y_t = W_t + aW_{t-1}/b - W_{t-2}/b
   \]

   Find the auto-covariance function of the two processes. Further show that they share the same autocorrelation function.

2. Let \( Z_t, \ t = 0, \pm 1, \pm 2, \ldots \), be independent normal random variables each with mean 0 and variance \( \sigma^2 \) and let \( a, b, c \) be constants. Which, if any, of the following processes are stationary? For each stationary process specify the mean and auto covariance function. If not stationary, tell the reason.

   (a) \( X_t = a + bZ_t + cZ_{t-1} \)
   (b) \( X_t = a + bZ_0 \)
   (c) \( X_t = Z_1 \cos(ct) + Z_2 \sin(ct) \)
   (d) \( X_t = Z_0 \cos(ct) \)
   (e) \( X_t = Z_t \cos(ct) + Z_{t-1} \sin(ct) \)
   (f) \( X_t = Z_tZ_{t-1} \)

3. Find the mean and auto covariance function of the ARMA(2,1) process,

   \[
   X_t = 2 + 1.3X_{t-1} - .4X_{t-2} + W_t + W_{t-1}, \quad \{W_t\} \sim \text{WN}(0, \sigma^2).
   \]

   Is this process causal or invertible?

4. Let \( \{X_t\} \) be the ARMA(1,1) process,

   \[
   X_t - \phi X_{t-1} = W_t + \theta W_t,
   \]

   where \( |\phi| < 1 \) and \( |\theta| < 1 \). Determine the coefficients \( \{\psi_j\} \) such that

   \[
   X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j},
   \]
and show that the autocorrelation function of \( \{X_t\} \) is given by
\[
\rho_X(1) = (1 + \phi\theta)(\phi + \theta)/(1 + \theta^2 + 2\phi\theta), \quad \rho_X(h) = \phi^{h-1}\rho(1) \text{ for } h \geq 1.
\]

5. Let \( \{X_t\} \) be the process
\[
X_t = A \cos(\pi t/3) + B \sin(\pi t/3) + W_t + 2.5W_{t-1}, \quad W_t \sim \text{WN}(0, \sigma^2)
\]
where \( A \) and \( B \) are uncorrelated random variables with mean 0 and variance \( \nu^2 \), both of which are further uncorrelated with \( \{W_t\} \). Find the autocovariance function and the spectral distribution of \( \{X_t\} \).

6. Suppose that \( \{X_t\} \) is a zero-mean stationary process with ACVF \( \gamma_X(h) \) and corresponding ACF \( \rho_X(h) \).
   
   (a) Under the assumption that \( \rho_X(2) \neq \pm 1 \), show that the best linear predictor of \( X_2 \) based on \( X_1 \) and \( X_3 \) is
   
   \[
   \hat{X}_2 = \frac{\rho_X(1)}{1 + \rho_X(2)}(X_1 + X_3)
   \]
   
   and has a mean square error (MSE) given by
   
   \[
   E(X_2 - \hat{X}_2)^2 = \gamma_X(0) \left( 1 - \frac{2\rho_X^2(1)}{1 + \rho_X(2)} \right)
   \]

   (b) A second way of predicting \( X_2 \) based on \( X_1 \) and \( X_3 \) is linear interpolation:
   
   \[
   \tilde{X}_2 = \frac{X_1 + X_3}{2}.
   \]
   
   Determine the mean square error for this predictor.

   (c) If \( \{X_t\} \) is an AR(1) process: i.e., \( X_t - \phi X_{t-1} = W_t \) with \( |\phi| < 1 \). Compare the ratio of the MSE of \( \tilde{X}_2 \) to the MSE for \( \hat{X}_2 \) as \( \phi \to 1 \) and also as \( \phi \to -1 \) and write down your comments.

7. Let \( \{Z_t\} \) be IID \( N(0, \sigma^2) \) noise, and define
\[
X_t = Z_t
\]
where \( t \) is even and
\[
X_t = Z^2_{t-1}/(\sigma\sqrt{2}) - \sigma/\sqrt{2}
\]
where \( t \) is odd.

   (a) Show that \( \{X_t\} \) is WN(0, \( \sigma^2 \))

   (b) What is the best linear predictor of \( X_{n+1} \) given \( X_n \), and what is its associated mean square error?
(c) What is the best predictor of $X_{n+1}$ given $X_n$, and what is its associated mean square error?

(d) Compare your results to part (2) with part (3). Comments?

8. (Bonus). Check Proposition 5.1 of the lecture notes. Find a stationary processes $\{X_t\}$ such that $\Gamma_n$ is positive definite for every $n$ and $\gamma_X(0) > 0$, but $\gamma_X(h)$ does not converges to 0 as $h \to \infty$. 