HW 3 (Due April. 15, 2015)

1. Show that if \( n > p \), the likelihood of the observations \( \{X_1, \ldots, X_n\} \) of the causal AR\((p)\) process,

\[
X_t - \phi_1X_{t-1} - \cdots - \phi_pX_{t-p} = W_t, \quad W_t \sim N(0, \sigma^2),
\]

is

\[
L(\phi, \sigma^2) = (2\pi\sigma^2)^{-n/2}(\det G_p)^{-1/2}
\times \exp \left\{ -\frac{1}{2\sigma^2} \left[ X_p'G_p^{-1}X_p + \sum_{t=p+1}^{n} (X_t - \phi_1X_{t-1} - \cdots - \phi_pX_{t-p})^2 \right] \right\}
\]

where \( X_p = (X_1, \ldots, X_p)' \) and \( G_p = \sigma^{-2}E(X_pX_p') \). Further given two observations \( x_1 \) and \( x_2 \) from the causal AR\((1)\) process,

\[
X_t - \phi X_{t-1} = W_t, \quad W_t \sim N(0, \sigma^2)
\]

such that \(|x_1| \neq |x_2|\), find the MLE of \( \phi \) and \( \sigma^2 \).

2. For an AR\((p)\) process, show that \( \det \Gamma_m = (\det \Gamma_p)\sigma^{2(m-p)} \) for all \( m > p \). Conclude that the \((m, m)\) component of \( \Gamma_m^{-1} \) is \( (\det \Gamma_{m-1})/(\det \Gamma_m) = \sigma^{-2} \). (This proves the \( \sqrt{n}\hat{\phi}_{mm} \to^d N(0, 1) \) in Theorem 6.2)

3. Suppose that \( \{X_t\} \) is an ARIMA\((p, d, q)\) process, satisfying the difference equations

\[
\phi(B)(1 - B)^dX_t = \theta(B)W_t, \quad W_t \sim WN(0, \sigma^2)
\]

show that these difference equations are also satisfied by the process

\[
Z_t = X_t + \alpha_0 + \alpha_1t + \cdots + \alpha_{d-1}t^{d-1}
\]

for arbitrary random variables \( \alpha_0, \alpha_1, \ldots, \alpha_{d-1} \).

4. Let \( \{X_t\} \) be the seasonal process,

\[
(1 - .7B^2)X_t = (1 + .3B^2)W_t, \quad W_t \sim WN(0, \sigma^2)
\]

Find the coefficients \( \{\psi_j\} \) such that \( X_t = \sum_{j=0}^{\infty} \psi_jW_{t-j} \). Further, find the coefficients \( \{\pi_j\} \) such that \( W_t = \sum_{j=0}^{\infty} \pi_jX_{t-j} \).