Name: Key

STAT 520 – Test 1 – Fall 2025

1) Consider a time series process $\{Y_t\}$ to be a process defined as:
$Y_t = (e_1 + \ldots + e_t) / t$. Here, $e_1, \ldots, e_t$ are iid random variables, each with mean zero and variance σ_e^2. It can be shown (*you do not need to show this*) that

$E(Y_t) = 0$ for all t, $\text{var}(Y_t) = \sigma_e^2 / t$, and for any $0 < t < s$, $\text{cov}(Y_t, Y_s) = \sigma_e^2 / s$.

(a) Is $\{Y_t\}$ a weakly stationary time series process? Carefully explain your reasoning.

No. var(Y_t) depends on time t. And cov(Y_t, Y_s) depends on time s.

(b) Find corr(Y_4, Y_9), showing your work.

$$\text{corr}(Y_4, Y_9) = \frac{\text{cov}(Y_4, Y_9)}{\sqrt{\text{var}(Y_4)\,\text{var}(Y_9)}} = \frac{\sigma_e^2/9}{\sqrt{\left(\frac{\sigma_e^2}{4}\right)\left(\frac{\sigma_e^2}{9}\right)}} = \frac{1/9}{1/6} = \frac{6}{9} \approx 0.67$$

(c) Note that corr(Y_1, Y_4) = 0.5 for this model. Is this a stronger or weaker correlation than the correlation you found in part (b)? Weaker

2) Suppose X and Y are random variables, with var(X) = 1, var(Y) = 3, and cov(X, Y) = –1.

(a) Find var(X + 2Y), showing your work.

$$\text{var}(X+2Y) = \text{var}(X) + \text{var}(2Y) + 2\,\text{cov}(X, 2Y)$$
$$= \text{var}(X) + 4\,\text{var}(Y) + 2(2)\,\text{cov}(X,Y) = 1 + 4(3) - 4 = \boxed{9}$$

(b) Find cov(X, X + 2Y), showing your work.

$$\text{cov}(X, X+2Y) = \text{cov}(X,X) + 2\,\text{cov}(X,Y)$$
$$= 1 - 2 = \boxed{-1}$$

(c) Find corr(X, X + 2Y), showing your work.

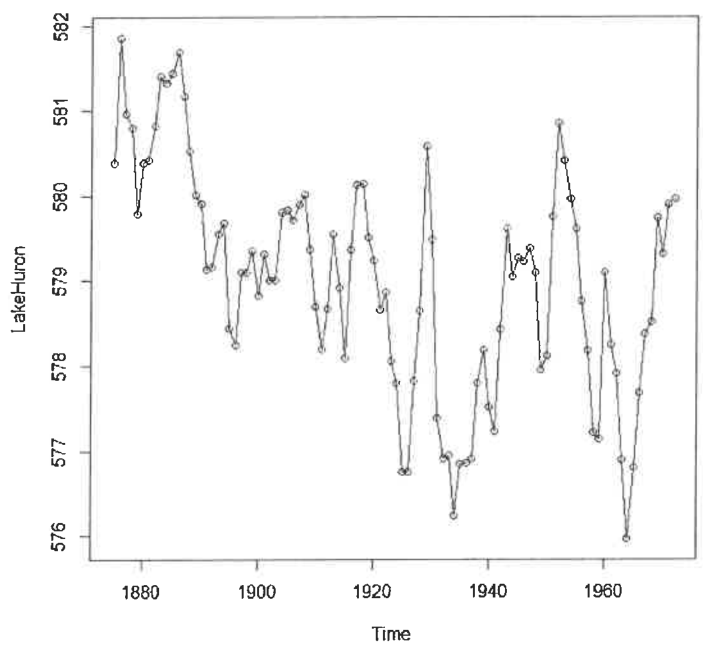
$$\text{corr}(X, X+2Y) = \frac{\text{cov}(X, X+2Y)}{\sqrt{\text{var}(X)\,\text{var}(X+2Y)}} = \frac{-1}{\sqrt{(1)(9)}}$$

$$= -\frac{1}{3} \approx -0.33$$

3) A student wrote the following statement: "If the mean function of a time series process is constant over time, then that process is stationary." Is this statement correct or wrong? Briefly explain why.

This is not correct — this only involves one condition for (weak) stationarity. The variance function must be constant over time, and the autocovariance must depend only on the lag, not on time.

4) The water level of Lake Huron was measured for 98 consecutive years. A plot of the time series is given below.



a) The analyst decided to fit a linear trend model to this time series. Briefly discuss whether you do or do not agree with this choice, based on this initial look at the data.

Answers may vary, but should reflect trends/patterns in the plot.

b) Partial R output from the linear trend model fit is given in the next page. Write the equation of the estimated linear trend model:

$$\hat{\mu}_t = 625.555 - 0.0242\,t$$

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lm(formula = LakeHuron ~ time(LakeHuron))

Coefficients:
                   Estimate Std. Error t value Pr(>|t|)
(Intercept)      625.554918   7.764293  80.568  < 2e-16 ***
time(LakeHuron)   -0.024201   0.004036  -5.996 3.55e-08 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.13 on 96 degrees of freedom
Multiple R-squared:  0.2725,    Adjusted R-squared:  0.2649
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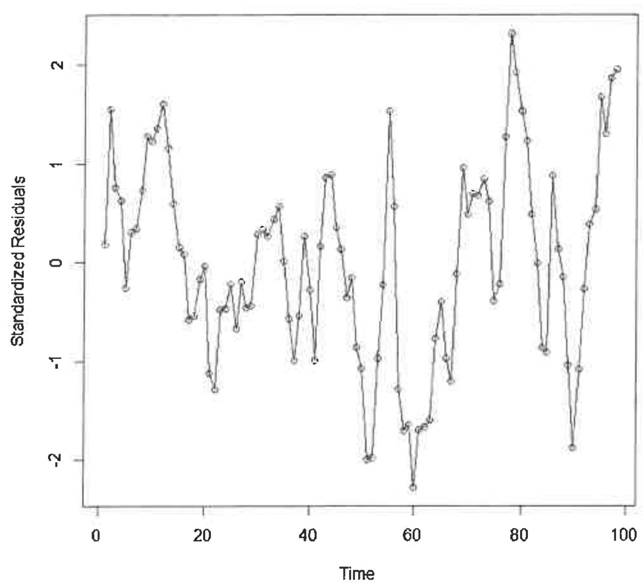
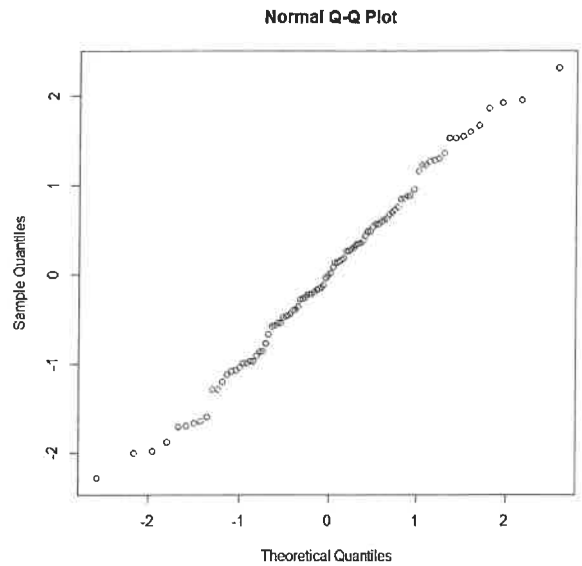
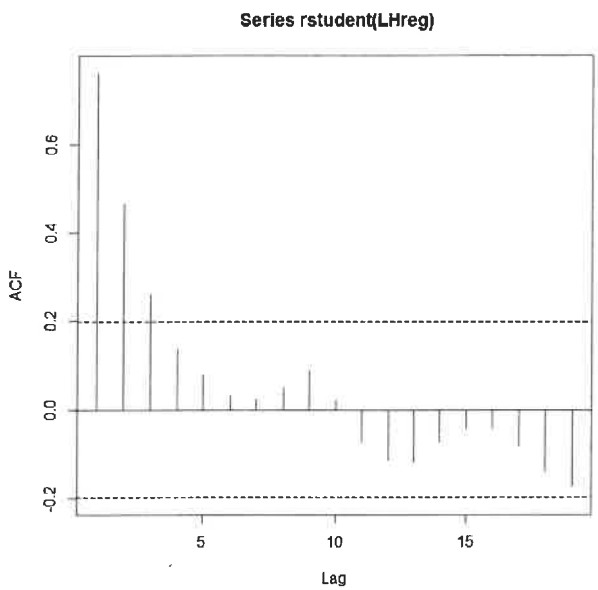
c) Various plots (three in all) involving the standardized residuals for this linear trend model are given below. For each plot, write a brief comment explaining what can be concluded about the stochastic component of the model, based on the respective plot.



- Possibly increasing variance over time
- The residuals seem to "hang together", a sign of positive autocorrelation.



- The residuals appear normal based on the straight Q-Q plot.



- This does not look like white noise.
- There is positive autocorrelation at the first two or three lags.

d) Do you believe the observed number of runs for the series of standardized residuals would be less than, greater than, or approximately equal to the expected number of runs under the assumption of independence? Briefly explain your answer.

Less than, since the ACF plot shows positive autocorrelation.

e) The BIC for the linear time trend model for the Lake Huron data was found to be 313.9. The BIC for a quadratic time trend model for the same data was found to be 298.2. What is your conclusion from these two values? The quadratic model is preferred since its BIC is lower.

5) Suppose $\{e_t\}$ is a white noise process with mean zero and variance σ_e^2. Let $\{Y_t\}$ be a process defined as:

$Y_t = e_t - 0.3e_{t-2}$.

a) Find $E(Y_t)$, $var(Y_t)$, $cov(Y_t, Y_{t-1})$, $cov(Y_t, Y_{t-2})$, and $cov(Y_t, Y_{t-3})$. Show your work.

$$E(Y_t) = 0 - 0.3(0) = 0$$

$$var(Y_t) = var(e_t) + 0.09\, var(e_{t-2})$$
$$= \sigma_e^2 + 0.09\,\sigma_e^2 = 1.09\,\sigma_e^2$$

$$cov(Y_t, Y_{t-1}) = cov(e_t - 0.3e_{t-2},\, e_{t-1} - 0.3e_{t-3})$$
$= 0$ since there are no overlapping terms

$$cov(Y_t, Y_{t-2}) = cov(e_t - 0.3e_{t-2},\, e_{t-2} - 0.3e_{t-4})$$
$$= 0 - 0 - 0.3\,cov(e_{t-2}, e_{t-2}) - 0$$
$$= -0.3\,\sigma_e^2$$

$$cov(Y_t, Y_{t-3}) = cov(e_t - 0.3e_{t-2},\, e_{t-3} - 0.3e_{t-5})$$
$= 0$ since no overlapping terms

(b) Find $corr(Y_t, Y_{t-1})$, $corr(Y_t, Y_{t-2})$, and $corr(Y_t, Y_{t-3})$. Show your work.

$$corr(Y_t, Y_{t-1}) = \frac{cov(Y_t, Y_{t-1})}{\sqrt{var(Y_t)\,var(Y_{t-1})}} = 0$$

$$corr(Y_t, Y_{t-2}) = \frac{cov(Y_t, Y_{t-2})}{\sqrt{var(Y_t)\,var(Y_{t-2})}} = \frac{-0.3\sigma_e^2}{\sqrt{(1.09\sigma_e^2)(1.09\sigma_e^2)}} = \frac{-0.3}{1.09} = -0.275$$

$$corr(Y_t, Y_{t-3}) = \frac{0}{\sqrt{(1.09\sigma_e^2)(1.09\sigma_e^2)}} = 0$$

(c) Describe the autocorrelation function of $\{Y_t\}$ as a piecewise function of the lags 0, 1, 2, 3, …

$$\rho_k = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{if } k=1 \\ -0.275 & \text{if } k=2 \\ 0 & \text{if } k \geq 3 \end{cases}$$

(d) Is $\{Y_t\}$ a weakly stationary time series process? Carefully explain your reasoning.

Yes. $E(Y_t)$ does not depend on t

$var(Y_t)$ does not depend on t.

$cov(Y_t, Y_{t-k})$ depends only on the lag k and not on t.

6) A trend model for periodic time series data, which includes sine and cosine terms, is called a

harmonic regression model.

Extra Credit: Who is the Japanese statistician who invented an Information Criterion that is commonly used for model selection?

(A) Thomas Bayes (B) Hirotugu Akaike (C) Genichi Taguchi (D) Motosaburo Masuyama

AIC