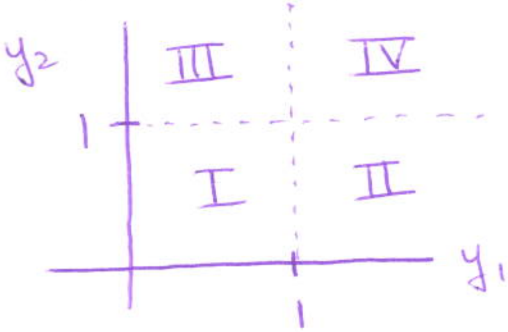
Example 4: Suppose the random vector (Y_1, Y_2) has joint pdf

$$f(y_1, y_2) = \begin{cases} y_1 + y_2 & \text{for } 0<y_1<1,\ 0<y_2<1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the joint cdf.

- Consider the diagram:



First, if $y_1 \le 0$ or $y_2 \le 0$, $F(y_1, y_2) = 0$ clearly.

If $0 < y_1 < 1$ and $0 < y_2 < 1$ (Region I):

$$F(y_1, y_2) = \int_0^{y_2} \int_0^{y_1} (t_1 + t_2)\, dt_1\, dt_2$$

$$= \int_0^{y_2} \left[\frac{t_1^2}{2} + t_1 t_2\right]_{t_1=0}^{y_1} dt_2 = \int_0^{y_2} \left(\frac{y_1^2}{2} + y_1 t_2\right) dt_2$$

$$= \left[\frac{y_1^2}{2} t_2 + \frac{y_1 t_2^2}{2}\right]_{t_2=0}^{y_2} = \frac{y_1^2 y_2}{2} + \frac{y_1 y_2^2}{2}$$

$$= \frac{1}{2} y_1 y_2 (y_1 + y_2)$$

If $y_1 \geq 1$ and $0 < y_2 < 1$ (Region II):

$$F(y_1, y_2) = \int_0^{y_2}\int_0^1 (t_1+t_2)\,dt_1\,dt_2 = \int_0^{y_2}\left[\frac{t_1^2}{2} + t_1 t_2\right]_{t_1=0}^{1} dt_2$$

$$= \int_0^{y_2}\left(\frac{1}{2}+t_2\right)dt_2 = \left[\frac{1}{2}t_2 + \frac{t_2^2}{2}\right]_0^{y_2} = \frac{y_2}{2} + \frac{y_2^2}{2} = \frac{1}{2}y_2(y_2+1)$$

Similarly, if $0 < y_1 < 1$ and $y_2 \geq 1$ (Region III):

$$F(y_1, y_2) = \int_0^1\int_0^{y_1} (t_1+t_2)\,dt_1\,dt_2 = \frac{1}{2}y_1(y_1+1)$$

And for $y_1 \geq 1$ and $y_2 \geq 1$ (Region IV):

$$F(y_1, y_2) = \int_0^1\int_0^1 (t_1+t_2)\,dt_1\,dt_2 = 1$$

So

$$F(y_1,y_2) = \begin{cases} 0 & \text{for } y_1 \leq 0 \text{ or } y_2 \leq 0 \\ \frac{1}{2}y_1 y_2 (y_1+y_2) & \text{for } 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1 \\ \frac{1}{2}y_2(y_2+1) & \text{for } y_1 \geq 1,\ 0 < y_2 < 1 \\ \frac{1}{2}y_1(y_1+1) & \text{for } 0 < y_1 < 1,\ y_2 \geq 0 \\ 1 & \text{for } y_1 \geq 1,\ y_2 \geq 1 \end{cases}$$

Finding Probabilities with Joint pdf's

- With bivariate distributions, we should <u>always</u> sketch the region of support in the (y_1, y_2) plane.

- Finding a probability amounts to integrating the joint pdf over a particular region of that support.