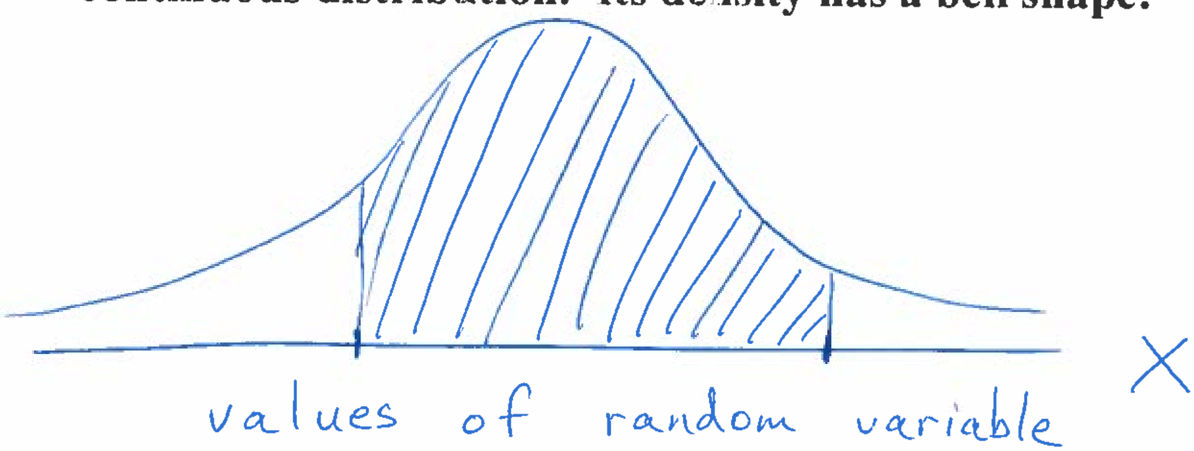
STAT 515 -- Chapter 5: Continuous Distributions

Probability distributions are used a bit differently for continuous r.v.'s than for discrete r.v.'s.

Continuous distributions typically are represented by a <u>probability density function</u> (pdf), or "density curve." (kind of a "theoretical" histogram)

A <u>density</u> curve is a representation of the <u>underlying population distribution</u> (not a description of actual sample data).

The <u>normal distribution</u> is a particular type of continuous distribution. Its density has a bell shape:



Properties of Density Functions:

(1) Density function always on or above the horizontal axis (curve can never have a negative value)

(2) Total area beneath the curve (between curve and horizontal axis) is exactly 1.

(3) An area under a density function represents a probability about the r.v. (or the proportion of observations we expect to have certain values).

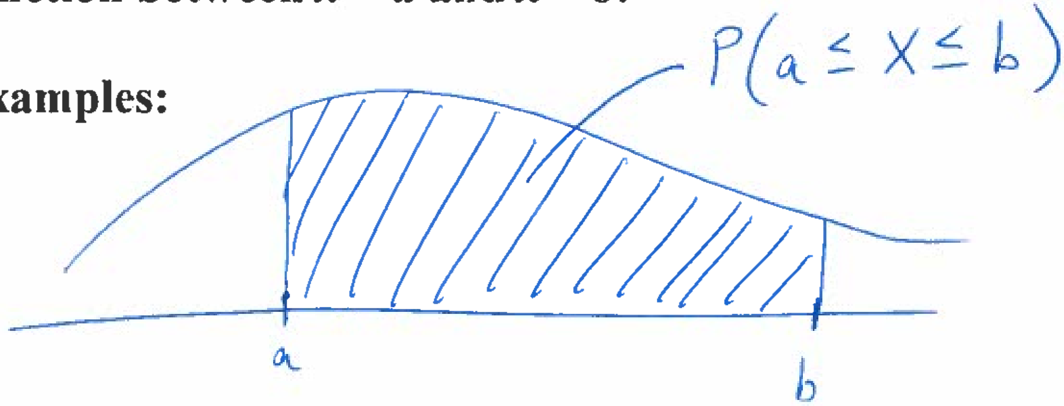
With discrete r.v.'s we looked at probability function (table, graph) to find probability of the r.v. taking a particular value.

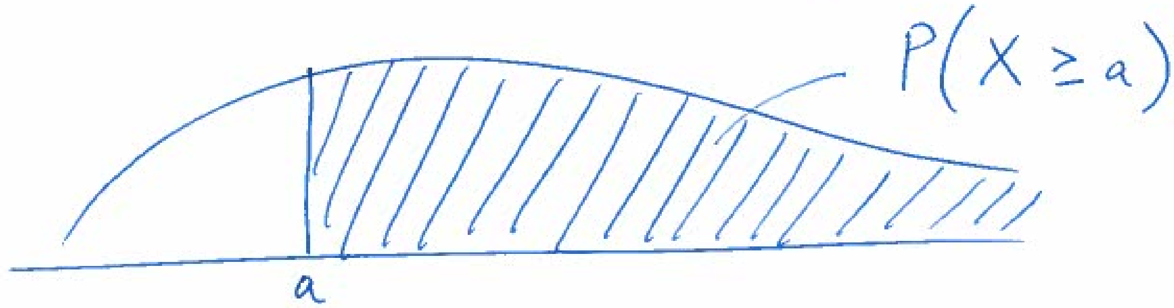
For continuous r.v.'s, the probability distribution will give us the probability that a value falls in an <u>interval</u> (for example, between two numbers).

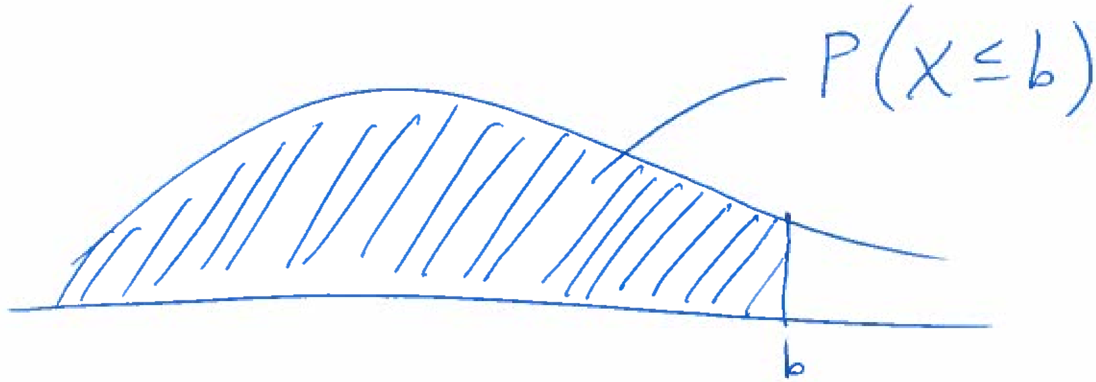
That is, the probability distribution of a continuous r.v. X will tell us $P(a \leq X \leq b)$, where a and b are particular numbers.

Specifically, $P(a \leq X \leq b)$ is the area under the density function between $x = a$ and $x = b$.

Examples:





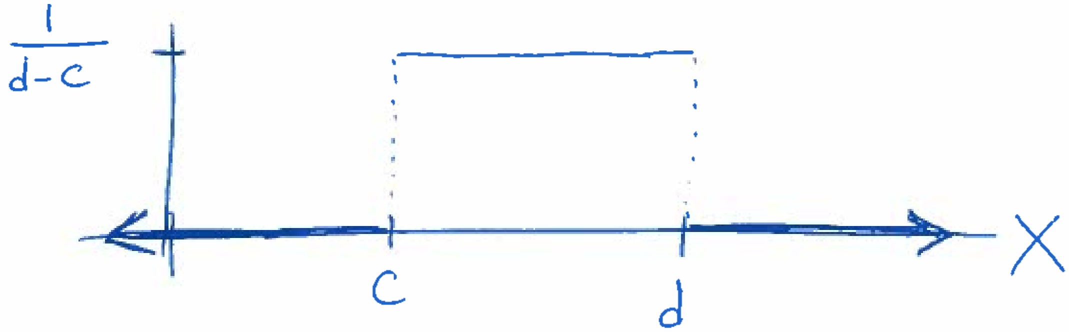


The Uniform Distribution

This is a simple example of a continuous distribution.

A uniform r.v. is equally likely to take any value between its lower limit (some number *c*) and its upper limit (some number *d*).

Density looks like a rectangle:



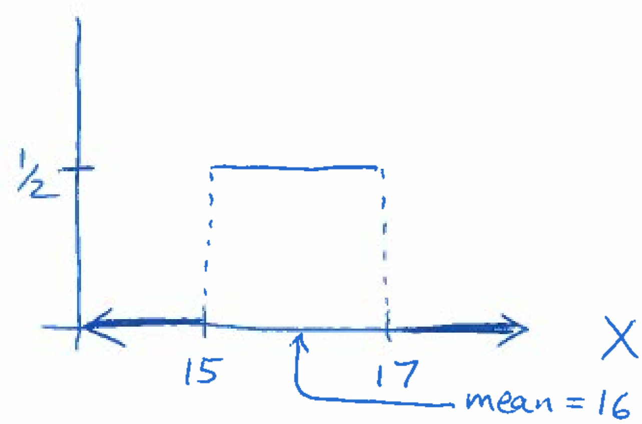
If total area is 1, then what is the height of the density function? $\frac{1}{d-c}$ since $(d-c)\left(\frac{1}{d-c}\right) = 1$.

- **Mean of a Uniform(*c*, *d*) r.v. = (*c* + *d*) / 2**
- **Std. deviation of a Uniform(*c*, *d*) r.v. = (*d* – *c*) / $\sqrt{12}$**

Example: A machine designed to fill 16-ounce water bottles actually dispenses a random amount between 15.0 and 17.0 ounces. The amount X of water dispensed is a Uniform(15, 17) random variable:

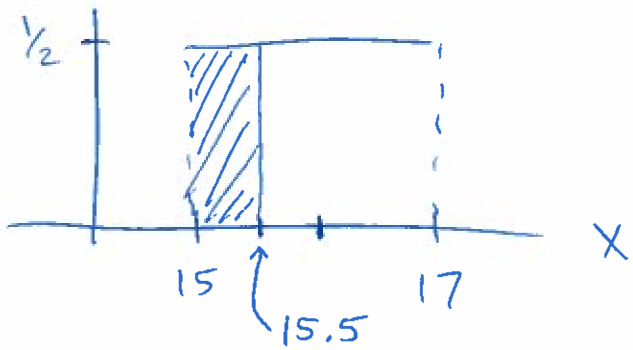
Density: $\frac{1}{17-15} = \frac{1}{2}$

s.d. = $\frac{17-15}{\sqrt{12}} \approx 0.58$



What is the probability that the bottle has less than 15.5 ounces of water?

P(X < 15.5) = P(15 < X < 15.5) =



$$\text{Area} = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \tfrac{1}{4} = 0.25$$

In general:

For X ~ Uniform(c, d): For $c \le a \le b \le d$

Above example:

P(a < X < b) = $\dfrac{b-a}{d-c} = \dfrac{15.5-15}{17-15} = \dfrac{0.5}{2} = \boxed{0.25}$

Probability that X is between 15.7 and 16.5 oz.?

$$P(15.7 < X < 16.5) = \frac{16.5-15.7}{17-15} = \frac{0.8}{2} = \boxed{0.4}$$

Probability that X is between 15.7 and 20 oz.?

$$P(15.7 < X < 20) = P(15.7 < X < 17) = \frac{17-15.7}{17-15}$$

$$= \frac{1.3}{2} = \boxed{0.65}$$

The Normal Distribution

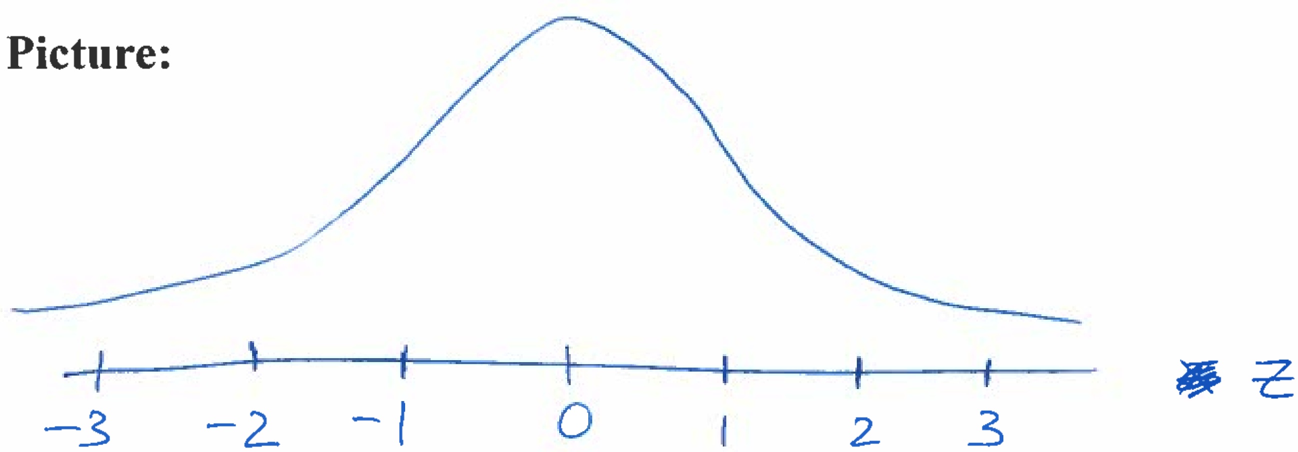
The density function for the normal distribution is complicated:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} \quad \textbf{for all } x$$

Note that the normal distribution changes depending on the values of the mean μ and the standard deviation σ.

<u>Standard Normal Distribution</u> [Notation: N(0, 1)]: The normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$.

Picture:



- **Mound-shaped, symmetric, centered at 0.**
- **Density always positive, even in "tails."**
- **Area under curve is 0.5 to left of zero, 0.5 to right of zero.**
- **Almost all area under curve (99.7%) between -3 and 3.**

Note: N(0, 1) distribution sometimes called the "z-distribution" and standard normal values are denoted by z.

Table II in back of book gives areas between 0 and certain listed values of z.

Example: Area under N(0, 1) curve between 0 and 1.24:

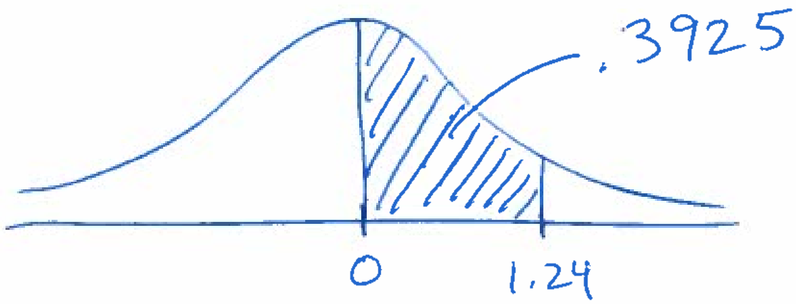


Table II: Go to row labeled 1.2, column labeled .04:
Correct area = 0.3925

What does this area mean?

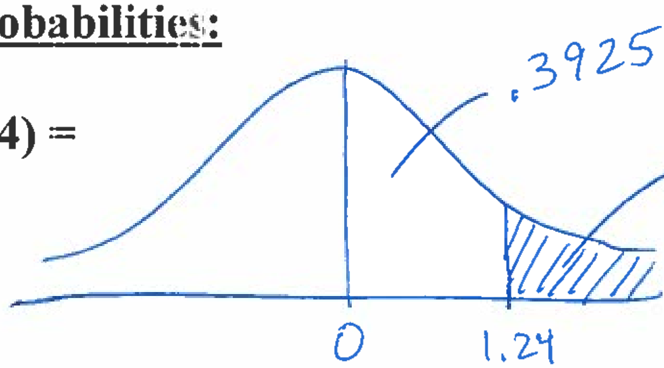
- **If *Z* is a r.v. with a standard normal distribution, then P(0 < *Z* < 1.24) =** 0.3925

[Note: Same as P(0 ≤ *Z* ≤ 1.24).]

- **We expect that 39.25% of the values of data having a standard normal distribution will be between 0 and 1.24**

<u>Other Probabilities:</u>

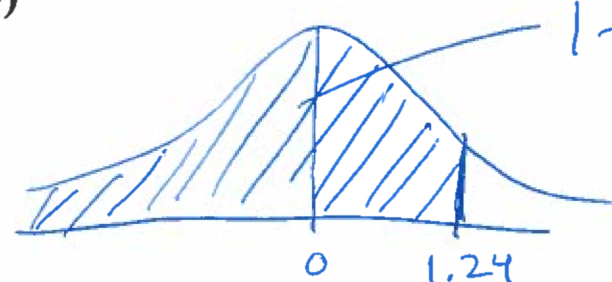
P(*Z* > 1.24) =



.5 − .3925 = $\boxed{.1075}$

P(*Z* < 1.24) =

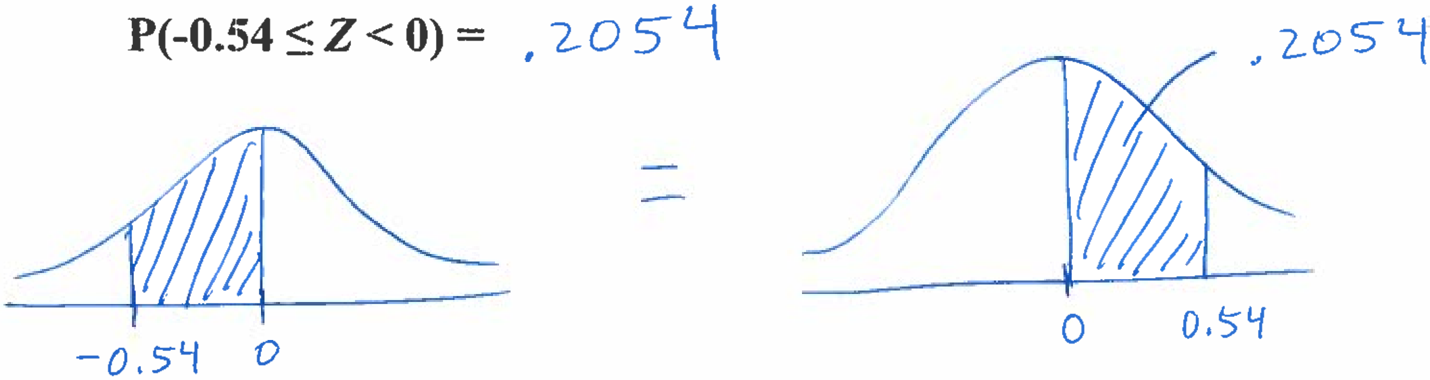
1 − .1075 = $\boxed{.8925}$



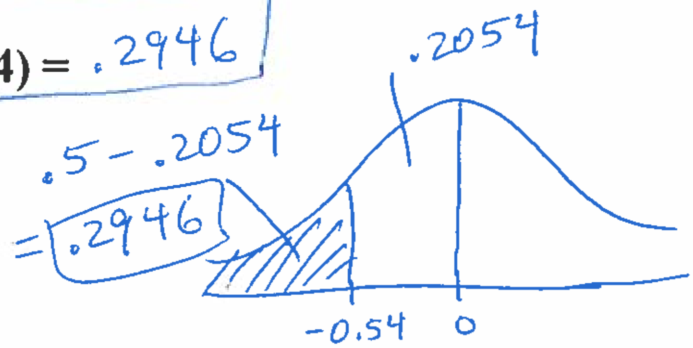
or: .5 + .3925 = $\boxed{.8925}$

Values to the left of zero? Use symmetry!

P(-0.54 ≤ *Z* < 0) = .2054



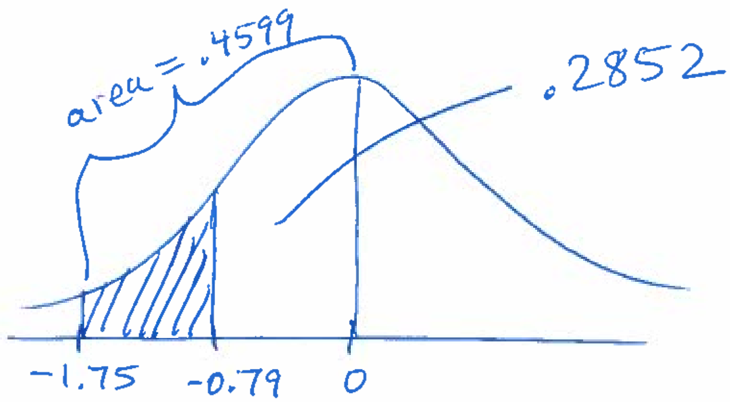
P(*Z* < -0.54) = .2946



P(-1.75 < *Z* < -0.79) =

.4599 - .2852

= $\boxed{.1747}$



P(-0.79 < *Z* < 1.16) =

.2852 + .3770

= $\boxed{.6622}$