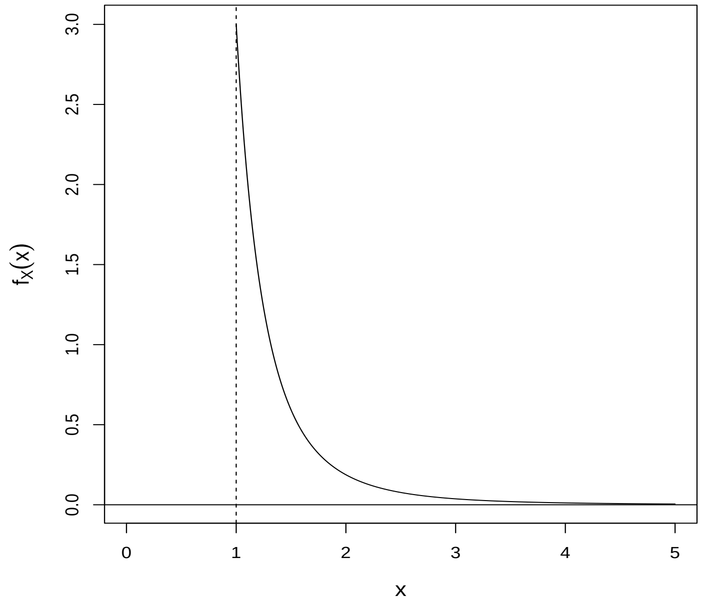
1. (a) Here is a graph of the pdf of X:



(b) We want

$$P(X > 2.5) = \int_{2.5}^{\infty} f_X(x)dx = \int_{2.5}^{\infty} \frac{3}{x^4}dx = 3\left(-\frac{1}{3}\right)\left(\frac{1}{x^3}\Big|_{2.5}^{\infty}\right) = -\left(0 - \frac{1}{2.5^3}\right) = 0.064.$$

This is the probability a particle exceeds 2.5 micrometers in size. This is the area under $f_X(x)$ to the right of 2.5.

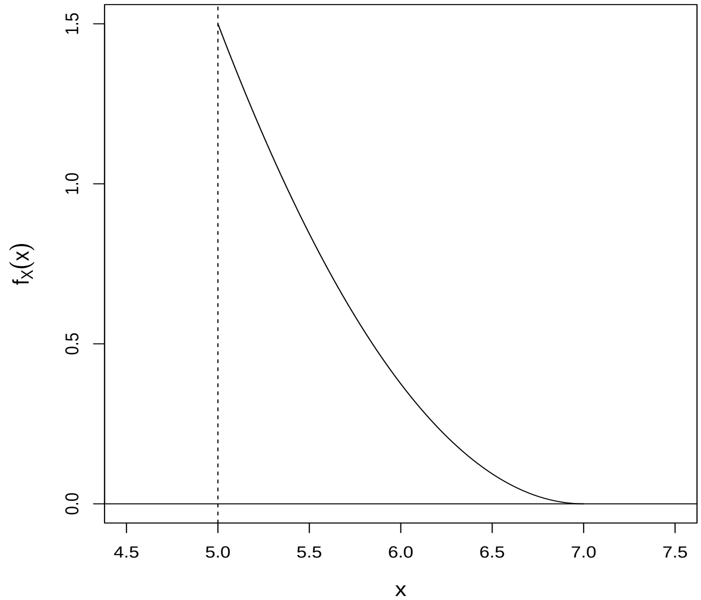
(b) We want

$$P(1.5 < X < 3) = F_X(3) - F_X(1.5) = \left(1 - \frac{1}{3^3}\right) - \left(1 - \frac{1}{1.5^3}\right) = \frac{1}{1.5^3} - \frac{1}{3^3} \approx 0.259.$$

Therefore, about 25.9% of the particles will be between 1.5 and 3 micrometers. This is the area under $f_X(x)$ between 1.5 and 3.

2. For part (a), a graph of the pdf of X is on the next page (top). For part (b), the mean of X is

$$\begin{aligned}
E(X) &= \int_5^7 x f_X(x)dx \\
&= \int_5^7 \frac{3}{8}x(7-x)^2 dx \\
&= \frac{3}{8}\int_5^7 x(49 - 14x + x^2)dx \\
&= \frac{3}{8}\int_5^7 (49x - 14x^2 + x^3)dx \\
&= \frac{3}{8}\left(\frac{49}{2}x^2 - \frac{14}{3}x^3 + \frac{1}{4}x^4\right)\Big|_5^7 \\
&= \frac{3}{8}\left\{\left[\frac{49}{2}(49) - \frac{14}{3}(343) + \frac{1}{4}(2401)\right] - \left[\frac{49}{2}(25) - \frac{14}{3}(125) + \frac{1}{4}(625)\right]\right\} = 5.5.
\end{aligned}$$



To find the variance of X, let's find $E(X^2)$ first:

$$\begin{aligned}
E(X^2) &= \int_5^7 x^2 f_X(x)dx \\
&= \int_5^7 \frac{3}{8}x^2(7-x)^2 dx \\
&= \frac{3}{8}\int_5^7 x(49-14x+x^2)dx \\
&= \frac{3}{8}\int_5^7 (49x^2-14x^3+x^4)dx \\
&= \frac{3}{8}\left(\frac{49}{3}x^3-\frac{14}{4}x^4+\frac{1}{5}x^5\right)\Big|_5^7 \\
&= \frac{3}{8}\left\{\left[\frac{49}{3}(343)-\frac{14}{4}(2401)+\frac{1}{5}(16807)\right]-\left[\frac{49}{3}(125)-\frac{14}{4}(625)+\frac{1}{5}(3125)\right]\right\} = 30.4.
\end{aligned}$$

From the variance computing formula, we have

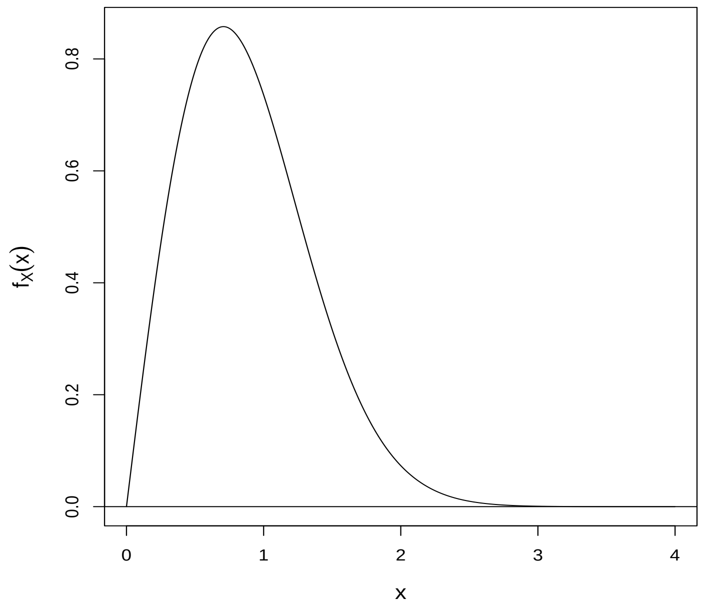
$$V(X) = E(X^2) - [E(X)]^2 = 30.4 - (5.5)^2 = 0.15.$$

I used R's `integrate` function to check my work:

```
> x.times.pdf = function(x){x*(3/8)*(7-x)^2}
> integrate(x.times.pdf,lower=5,upper=7) # E(X)
5.5 with absolute error < 6.1e-14

> x2.times.pdf = function(x){(x^2)*(3/8)*(7-x)^2}
> integrate(x2.times.pdf,lower=5,upper=7) # E(X^2)
30.4 with absolute error < 3.4e-13
```

3. (a) Here is a graph of the pdf of X:



(b) To find $\phi_{0.90}$, the 90th percentile, we set

$$P(X \leq \phi_{0.90}) = F_X(\phi_{0.90}) = 0.90$$

and solve for $\phi_{0.90}$. We have

$$\begin{aligned} 1 - e^{-\phi_{0.90}^2} = 0.90 &\implies e^{-\phi_{0.90}^2} = 0.10 \\ &\implies -\phi_{0.90}^2 = \ln(0.10) \\ &\implies \phi_{0.90}^2 = -\ln(0.10) \\ &\implies \phi_{0.90} = \sqrt{-\ln(0.10)} \approx 1.517. \end{aligned}$$

Therefore, the 90th percentile is approximately 1.517 years. This means that 90% of the transistors will fail before this time. Ten percent of the transistors' failure times will be larger than this.

4. A graph of the pdf of X is on the next page (top). For part (a), we have

$$P(X > 50) = \int_{50}^{\infty} f_X(x)dx = \int_{50}^{\infty} 0.03e^{-0.03x}dx.$$

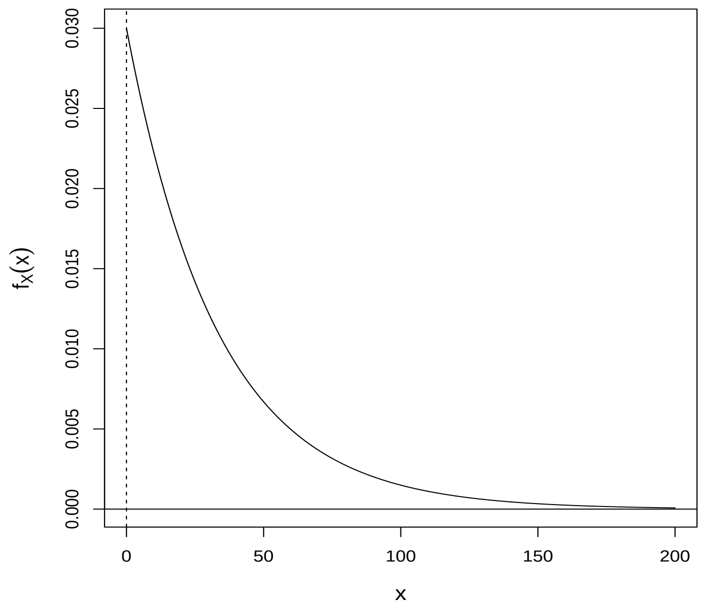
We could also write

$$P(X > 50) = 1 - P(X \leq 50) = 1 - F_X(50),$$

where $F_X(x)$ is the cdf of X. Recall $F_X(x) = 1 - e^{-\lambda x}$, for $X \sim \text{exponential}(\lambda)$. Therefore,

$$1 - F_X(50) = 1 - [1 - e^{-0.03(50)}] = e^{-1.5} \approx 0.223.$$

```
> options(digits=3)
> 1-pexp(50,0.03)
[1] 0.223
```



(b) The mean concentration is

$$E(X) = \frac{1}{\lambda} = \frac{1}{0.03} \approx 33.3 \text{ ppm}.$$

The median concentration $\phi_{0.50}$ solves

$$P(X \leq \phi_{0.50}) = F_X(\phi_{0.50}) = 0.50.$$

We have

$$\begin{aligned} 1 - e^{-0.03\phi_{0.50}} = 0.50 \quad &\Longrightarrow \quad e^{-0.03\phi_{0.50}} = 0.50 \\ &\Longrightarrow \quad -0.03\phi_{0.50} = \ln(0.50) \\ &\Longrightarrow \quad \phi_{0.50} = \frac{-\ln(0.50)}{0.03} \approx 23.1 \text{ ppm}. \end{aligned}$$

The mean $E(X)$ is larger because the pdf of X is skewed to the right side.

```
> qexp(0.5,0.03)
[1] 23.1
```

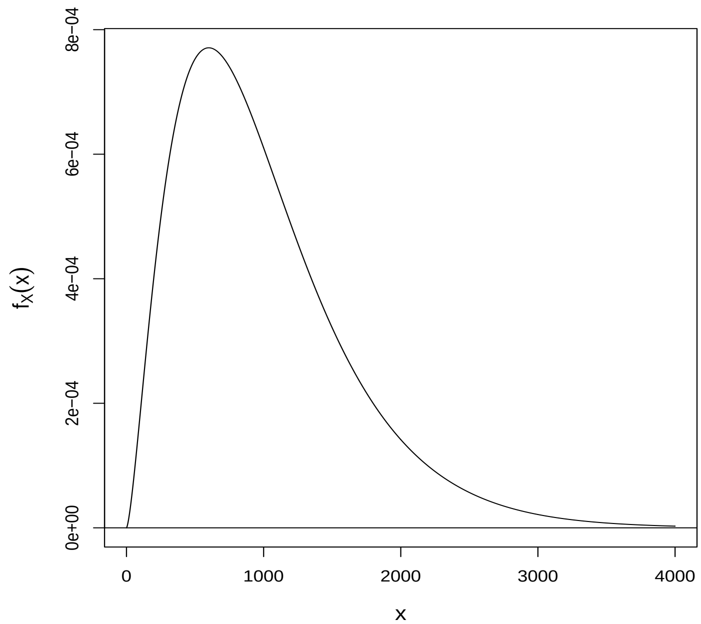
5. A graph of the pdf of X is on the next page (top). For part (a), we have

$$P(X < 365) = \int_0^{365} \frac{(0.0025)^{2.5}}{\Gamma(2.5)} \, x^{1.5} e^{-0.0025x} dx.$$

We have no choice but to use R here (unless you want to do this integral numerically):

```
> pgamma(365,2.5,0.0025)
[1] 0.127
```

Therefore, about 12.7% of the patients will die in less than one year.



(b) Think of each patient as a "trial," and dying in less than one year as a "success." If Y is the number of patients that die in less than one year (out of 10), then $Y \sim b(10, 0.127)$. We want

$$\begin{aligned} P(Y \geq 2) &= 1 - P(Y \leq 1) \\ &= 1 - P(Y = 0) - P(Y = 1) \\ &= 1 - \binom{10}{0}(0.127)^0(0.873)^{10} - \binom{10}{1}(0.127)^1(0.873)^9 \approx 0.369. \end{aligned}$$

Therefore, the probability at least 2 patients die within one year is approximately 0.369.

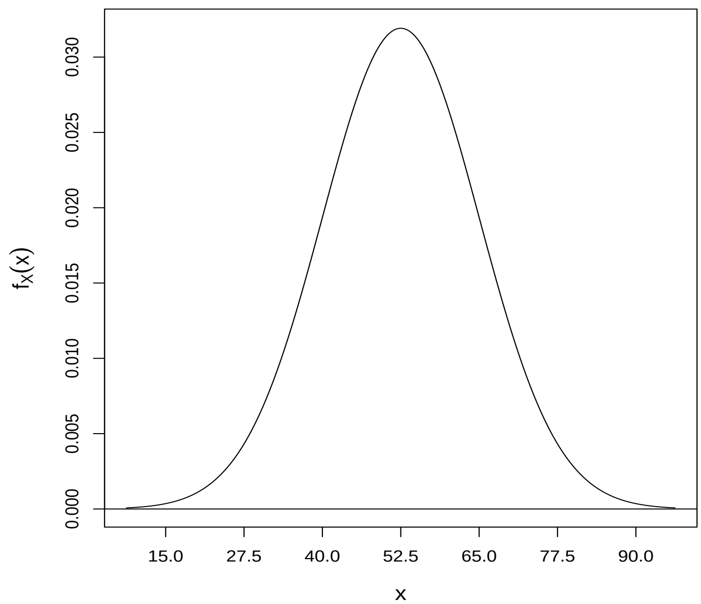
```
> 1-pbinom(1,10,0.127)
[1] 0.369
```

6. (a) Recall the relationship between the Poisson and the exponential and gamma distributions. The time until the first landing ("first occurrence") has an exponential distribution with $\lambda = 4.5$. The time until the third landing ("third occurrence") follows a gamma distribution with $\alpha = 3$ and $\lambda = 4.5$.
(b) Let X denote the number of landings (per hour) so that $X \sim \text{Poisson}(\lambda = 4.5)$. The probability there are no landings in any given hour is

$$P(X = 0) = \frac{4.5^0 e^{-4.5}}{0!} = e^{-4.5} \approx 0.011.$$

We can also get this answer using the exponential distribution. If there are no landings in a given hour, then the time until the first landing (exponential) must be larger than 1 hour. Therefore, if $T \sim \text{exponential}(\lambda = 4.5)$, we have

$$P(T > 1) = 1 - P(T \leq 1) = 1 - [1 - e^{-4.5(1)}] = e^{-4.5}.$$



7. A graph of the pdf of X is above. The tick marks on the horizontal axis are located at $\mu = 52.5$, $\mu \pm \sigma$, $\mu \pm 2\sigma$, and $\mu \pm 3\sigma$. For the parts below, we use the `pnorm` and `qnorm` functions in R.

For part (a), we want

$$P(X > 75) = 1 - P(X \leq 75) \approx 0.0359.$$

```
> 1-pnorm(75,52.5,12.5)
[1] 0.0359
```

(b) We want

$$P(40 < X < 65) = F_X(65) - F_X(40) \approx 0.683.$$

```
> pnorm(65,52.5,12.5)-pnorm(40,52.5,12.5)
[1] 0.683
```

It makes sense this answer is close to 68%. Note that 40 and 65 are both one standard deviation away from the mean $\mu = 52.5$.
(c) The 99th percentile is $\phi_{0.99} \approx 81.6$ ppm. This is calculated in R as follows:

```
> qnorm(0.99,52.5,12.5)
[1] 81.6
```

This means 99% of the daily PM10 levels (read by this sensor) will be less than 81.6 micrometers per m^3. Only 1% of the PM10 levels will be larger than this.

ADDITIONAL R CODE:

```
# Problem 1(a)
x = seq(1,5,0.01)
pdf = 3/x^4
plot(x,pdf,type="l",xlab="x",xlim=c(0,5),ylab=expression(f[X](x)),cex.lab=1.25)
abline(h=0)
abline(v=1,lty=2)


# Problem 2(a)
x = seq(5,7,0.005)
pdf = (3/8)*(7-x)^2
plot(x,pdf,type="l",xlab="x",xlim=c(4.5,7.5),ylab=expression(f[X](x)),cex.lab=1.25)
abline(h=0)
abline(v=5,lty=2)


# Problem 3(a)
x = seq(0,4,0.01)
pdf = 2*x*exp(-(x^2))
plot(x,pdf,type="l",xlab="x",xlim=c(0,4),ylab=expression(f[X](x)),cex.lab=1.25)
abline(h=0)


# Problem 4
x = seq(0,200,0.01)
pdf = dexp(x,0.03)
plot(x,pdf,type="l",lty=1,xlab="x",ylab=expression(f[X](x)),cex.lab=1.25)
abline(h=0)
abline(v=0,lty=2)


# Problem 5
x = seq(0,4000,1)
plot(x,dgamma(x,2.5,0.0025),type="l",lty=1,xlab="x",ylab=expression(f[X](x)),
    cex.lab=1.25)
abline(h=0)


# Problem 7
x = seq(8.75,96.25,0.05)
pdf = dnorm(x,52.5,12.5)
plot(x,pdf,type="l",xlab="x",ylab=expression(f[X](x)),xaxp=c(15,90,6),
    cex.lab=1.25)
abline(h=0)
```