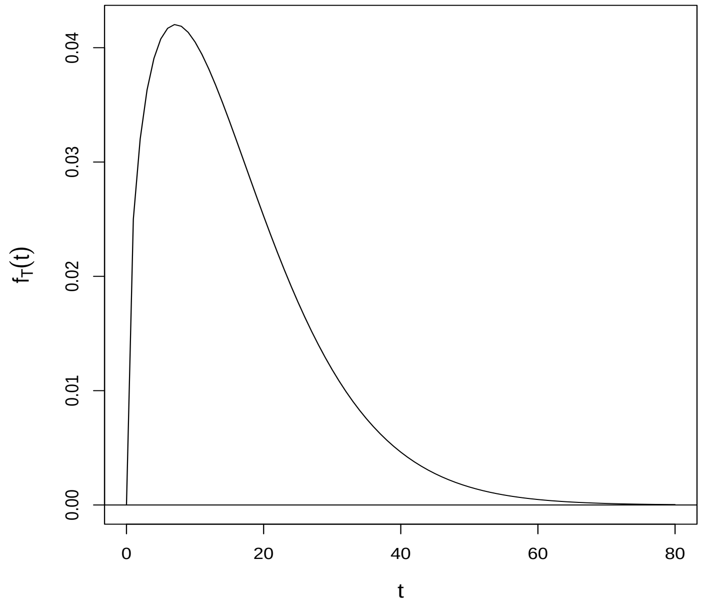
1. (a) Here is a graph of the pdf of T:



(b) The mean of T is

$$E(T) = \eta\Gamma\left(1+\frac{1}{\beta}\right) = 17.5\ \Gamma\left(1+\frac{1}{1.4}\right) = 17.5\ \Gamma\left(\frac{12}{7}\right) \approx 15.9.$$

The mean time to death is 15.9 months.

```
> options(digits=3)
> 17.5*gamma(12/7)
[1] 15.9
```

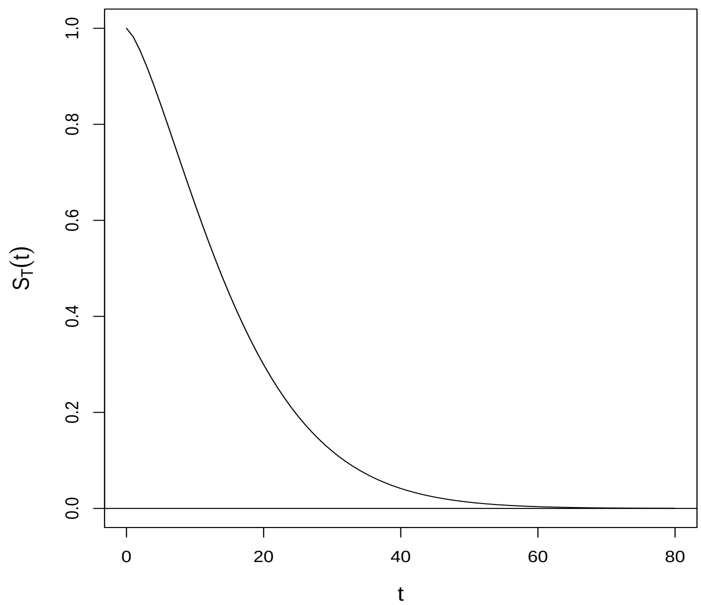
We can find the median $\phi_{0.5}$ by solving $F_T(\phi_{0.5}) = 0.5$ for $\phi_{0.5}$ like we did in the notes. We have

$$\begin{aligned}
F_T(\phi_{0.5}) = 1 - \exp\left[-\left(\frac{\phi_{0.5}}{17.5}\right)^{1.4}\right] = 0.5 \quad &\Longrightarrow \quad \exp\left[-\left(\frac{\phi_{0.5}}{17.5}\right)^{1.4}\right] = 0.5 \\
&\Longrightarrow \quad -\left(\frac{\phi_{0.5}}{17.5}\right)^{1.4} = \ln(0.5) \\
&\Longrightarrow \quad \left(\frac{\phi_{0.5}}{17.5}\right)^{1.4} = -\ln(0.5) \\
&\Longrightarrow \quad \frac{\phi_{0.5}}{17.5} = [-\ln(0.5)]^{1/1.4} \\
&\Longrightarrow \quad \phi_{0.5} = 17.5[-\ln(0.5)]^{1/1.4} \approx 13.5 \text{ months.}
\end{aligned}$$

We can also find the median $\phi_{0.5}$ by using the `qweibull` function in R:

```
> qweibull(0.5,1.4,17.5)
[1] 13.5
```

The median time to death is 13.5 months. This means 50% of this population of patients will die before 13.9 months. The mean is larger than the median because the pdf $f_T(t)$, shown above, is skewed to the right side. This type of skewness makes the mean larger.



(c) The survivor function is

$$S_T(t) = \begin{cases} 1, & t \le 0 \\ \exp\left[-\left(\frac{t}{17.5}\right)^{1.4}\right], & t > 0 \end{cases}.$$

This function is shown above. Note that

$$S_T(24) = \exp\left[-\left(\frac{24}{17.5}\right)^{1.4}\right] \approx 0.211.$$

This represents the proportion of the population of patients who are still alive at time $t = 24$ months.

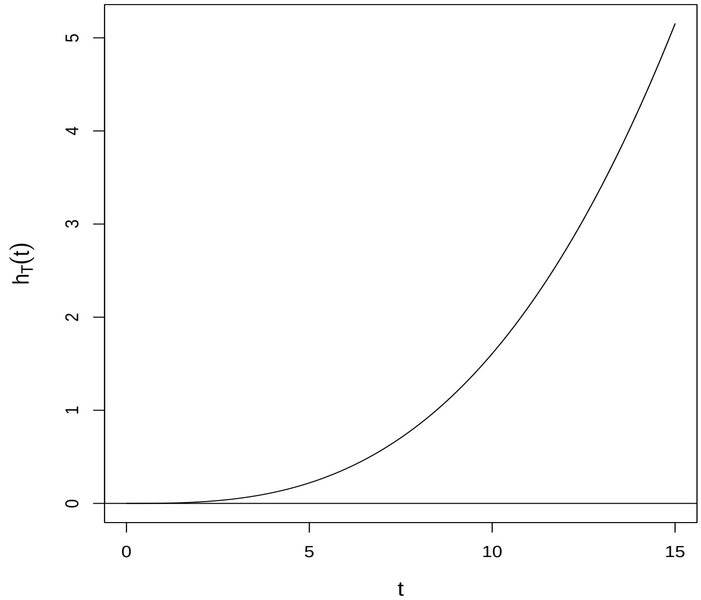
```
> 1-pweibull(24,1.4,17.5) # survivor function at t=24
[1] 0.211
```

(d) We want the 95th percentile $\phi_{0.95}$. We can find $\phi_{0.95}$ by solving $F_T(\phi_{0.95}) = 0.95$ for $\phi_{0.95}$ like we did in part (b) for the median.

It's easier to use R's `qweibull` function:

```
> qweibull(0.95,1.4,17.5)
[1] 38.3
```

This means 95% of this population of patients will die before 38.3 months.
(e) The hazard function $h_T(t)$ is increasing because the Weibull shape parameter $\beta > 1$. This means the rate of death (failure) is increasing over time for this population of patients. In other words, the population of patients is "getting weaker" over time.



2. We will find the maximum likelihood estimates using R like we did in Example 5.2 (in the course notes). Here is the code:

```
> library(fitdistrplus)
> options(digits=3)
> fitdist(turbo,distr="weibull",method="mle")
Fitting of the distribution ’ weibull ’ by maximum likelihood
Parameters:
      estimate Std. Error
shape     3.87      0.518
scale     6.92      0.295
```

The maximum likelihood estimates are

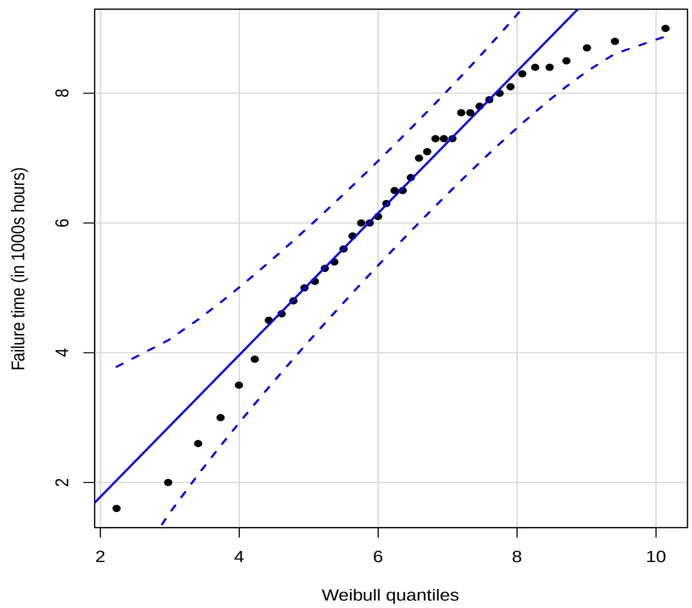
$$\begin{aligned} \widehat{\beta} &\approx 3.87 \\ \widehat{\eta} &\approx 6.92. \end{aligned}$$

(b) The estimated hazard function is

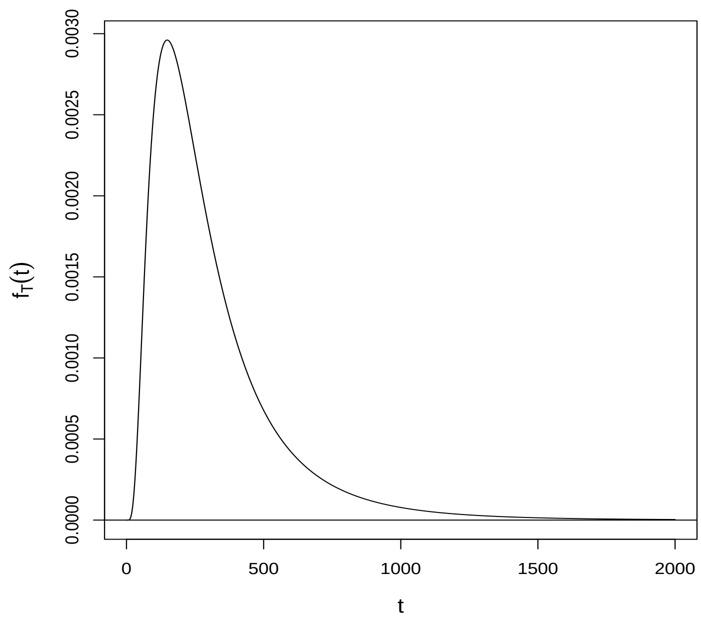
$$h_T(t) = \frac{3.87}{6.92}\left(\frac{t}{6.92}\right)^{3.87-1},$$

which is almost a cubic function of t (see figure above). We know the (estimated) hazard function $h_T(t)$ is increasing because $\widehat{\beta} > 1$. This means the rate of failure is increasing over time for this population of turbochargers. In other words, the population of turbochargers is "getting weaker" over time.

(c) The Weibull qq plot is shown at the top of the next page. We see general agreement between the turbocharger data and the Weibull quantiles. We can feel comfortable that the Weibull distribution is a reasonable lifetime distribution for the population of all turbochargers.



3. (a) Here is a graph of the pdf of T:



(b) The mean semiconductor laser lifetime is

$$E(T) = e^{5.5+0.5/2} = e^{5.75} \approx 314.2 \text{ hours.}$$

(c) We want $P(T > 1000)$. From R,

```
> options(digits=2)
> 1-plnorm(1000,5.5,sqrt(0.5))
[1] 0.023
```

Therefore, approximately 2.3% of all turbochargers will "survive" at least 1000 hours.

(d) We want the 10th percentile $\phi_{0.10}$. From R,

```
> options(digits=3)
> qlnorm(0.1,5.5,sqrt(0.5))
[1] 98.9
```

This means that 10% of the semiconductor laser will fail before 98.9 hours.
(e) Compare the lognormal hazard function

$$h_T(t) = \frac{\dfrac{1}{t\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{1}{2\sigma^2}\,(\ln t-\mu)^2\right\}}{1-F_Z\left(\dfrac{\ln t-\mu}{\sigma}\right)}$$

to the Weibull hazard function

$$h_T(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}.$$

The Weibull hazard function is much easier! In addition, we know the Weibull shape parameter β determines whether $h_T(t)$ is increasing, constant, or decreasing. Therefore, this one single number allows us to characterize the strength of the population over time. The lognormal hazard function does not offer this easy interpretation.
(f) If we use the wrong distribution to make reliability calculations (e.g., finding probabilities and quantiles, etc.), then our calculations could be off. This would produce incorrect assessments of reliability, which could have negative downstream effects. For example, warranty periods are designated based on product reliability. If our calculations are not correct (and maybe even way off), then we could designate a warranty period that is either much too long or too short. This is why qq plots are helpful. These plots allow you to assess whether a given distribution (e.g., Weibull, etc.) is reasonable based on the observed data.

ADDITIONAL R CODE:

```
# Problem 1(a)
t = seq(0,80,0.1)
pdf = dweibull(t,1.4,17.5)
plot(t,pdf,type="l",lty=1,xlab="t",ylab=expression(f[T](t)),cex.lab=1.25)
abline(h=0)
```

```
# Problem 1(c)
t = seq(0,80,0.1)
survivor = 1-pweibull(t,1.4,17.5)
plot(t,survivor,type="l",lty=1,xlab="t",ylab=expression(S[T](t)),cex.lab=1.25)
abline(h=0)
```

```
# Problem 2(a)
turbo = c(1.6,2.0,2.6,3.0,3.5,3.9,4.5,4.6,4.8,5.0,
    5.1,5.3,5.4,5.6,5.8,6.0,6.0,6.1,6.3,6.5,
    6.5,6.7,7.0,7.1,7.3,7.3,7.3,7.7,7.7,7.8,
    7.9,8.0,8.1,8.3,8.4,8.4,8.5,8.7,8.8,9.0)
library(fitdistrplus)
options(digits=3)
fitdist(turbo,distr="weibull",method="mle")


# Problem 2(b)
t = seq(0,15,0.01)
pdf = dweibull(t,3.87,6.92)
survivor = 1-pweibull(t,3.87,6.92)
plot(t,pdf/survivor,type="l",lty=1,xlab="t",ylab=expression(h[T](t)),
    cex.lab=1.25)
abline(h=0)


# Problem 2(c)
library(car)
qqPlot(turbo,distribution="weibull",shape=3.87,scale=6.92,
xlab="Weibull quantiles",ylab="Failure time (in 1000s hours)",pch=16,
envelope=list(border=TRUE,style="lines"),id=FALSE)


# Problem 3(a)
t = seq(0,2000,1)
pdf = dlnorm(t,5.5,sqrt(0.5))
plot(t,pdf,type="l",xlab="t",ylab=expression(f[T](t)),cex.lab=1.25)
abline(h=0)
```