1. (a) There are 36 different characters possible for each position in the password:

$$(\underline{36} \ \underline{36} \ \underline{36} \ \underline{36} \ \underline{36} \ \underline{36})$$

By the multiplication rule of counting, there are

$$36 \times 36 \times 36 \times 36 \times 36 \times 36 = 36^6 = 2,176,782,336$$

possible passwords.
(b) This means each password has the same probability (chance) of being selected.
(c) Thinking of selecting a password as a random experiment with sample space S, there are

$$n_S = 2176782336$$

possible passwords from part (a), each of which is equally likely as the selection is done at random. Therefore, we just need to count n_A, the number of ways A can occur, and use

$$P(A) = \frac{n_A}{n_S}.$$

For the event A to occur, we need to have 6 different letters chosen, for example, something like

$$(\underline{\text{a}} \ \underline{\text{f}} \ \underline{\text{x}} \ \underline{\text{m}} \ \underline{\text{t}} \ \underline{\text{u}}).$$

The ordering of the letters of the passwords is **important** (different orderings give different passwords), so

$$n_A = P_6^{26} = \frac{26!}{(26-6)!} = \frac{26!}{20!} = 26(25)(24)(23)(22)(21) = 165,765,600.$$

Assuming each password is equally likely, we have

$$P(A) = \frac{n_A}{n_S} = \frac{165765600}{2176782336} \approx 0.076.$$

(d) Here we need to count n_B, the number of ways B can occur, and use

$$P(B) = \frac{n_B}{n_S}.$$

For the event B to occur, we need to have a password with exactly 3 letters and exactly 3 numbers, for example, something like

$$(\underline{\text{m}} \ \underline{9} \ \underline{\text{s}} \ \underline{\text{s}} \ \underline{3} \ \underline{8}).$$

We can determine n_B using the multiplication rule of counting:

- there are $n_1 = \binom{6}{3} = 20$ ways to choose 3 positions to hold the 3 letters (then, the numbers automatically go in the other 3 positions)
- there are $n_2 = 26^3$ ways to select the 3 letters $(26 \times 26 \times 26)$
- there are $n_3 = 10^3$ ways to select the 3 numbers $(10 \times 10 \times 10)$.

Therefore,

$$n_B = n_1 \times n_2 \times n_3 = \binom{6}{3} \times 26^3 \times 10^3 = 351,520,000.$$

Assuming each password is equally likely, we have

$$P(B) = \frac{n_B}{n_S} = \frac{351520000}{2176782336} \approx 0.161.$$

2. (a) Define the events

$$\begin{aligned} A &= \{\text{small package used for shipping}\} \\ B &= \{\text{shipment damaged}\}. \end{aligned}$$

We are given $P(A) = 0.40$, $P(B|A) = 0.02$, and $P(B|A') = 0.01$.
(b) Use the Law of Total Probability. The probability a shipment is damaged is

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|A')P(A') \\ &= 0.02(0.40) + 0.01(0.60) \\ &= 0.014. \end{aligned}$$

Therefore, the probability a shipment is not damaged is

$$P(B') = 1 - P(B) = 1 - 0.014 = 0.986$$

by the complement rule. That is, 98.6% of the all shipments will not be damaged.
(c) In this part, we are only considering damaged shipments, so this means "B has occurred." We want $P(A|B)$. We can use Bayes' Rule to get this:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A')P(A')} = \frac{P(B|A)P(A)}{P(B)} = \frac{0.02(0.40)}{0.014} = \frac{8}{14} \approx 0.571.$$

That is, among all damaged shipments, about 57.1% of these use small packages.

3. (a) The random variable X is discrete because it has a finite number of possible values. We can list them out: 0, 1, 2, 3, 4, and 5.
(b) The mean number of specimens containing the pollutant is

$$\begin{aligned} E(X) &= \sum_{\text{all } x} x p_X(x) \\ &= 0(0.10) + 1(0.25) + 2(0.30) + 3(0.20) + 4(0.10) + 5(0.05) = 2.1. \end{aligned}$$

To find the variance of X, we can use the variance-computing formula:

$$V(X) = E(X^2) - [E(X)]^2.$$

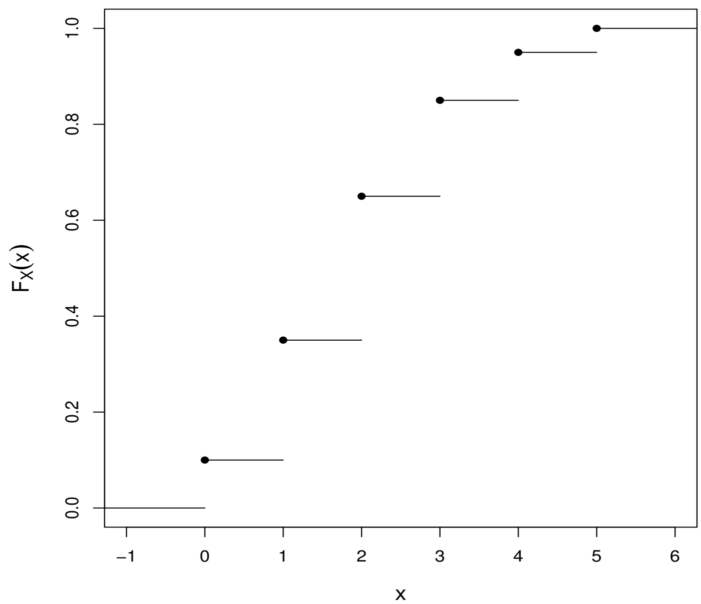
We know $E(X) = 2.1$. Also,

$$\begin{aligned} E(X^2) &= \sum_{\text{all } x} x p_X(x) \\ &= 0^2(0.10) + 1^2(0.25) + 2^2(0.30) + 3^2(0.20) + 4^2(0.10) + 5^2(0.05) = 6.1. \end{aligned}$$

Therefore,

$$V(X) = 6.1 - (2.1)^2 = 1.69.$$

(c) The cdf of X is easy to prepare by hand, but I used R. It is shown at the top of the next page.



4. (a) We want

$$\begin{aligned} P(X \geq 2) &= 1 - P(X \leq 1) \\ &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - \frac{(1.8)^0 e^{-1.8}}{0!} - \frac{(1.8)^1 e^{-1.8}}{1!} \\ &= 1 - e^{-1.8} - 1.8e^{-1.8} \approx 1 - 0.165 - 0.298 = 0.537. \end{aligned}$$

(b) Recall the Poisson-exponential and Poisson-gamma relationships:

- the time until the first asthma ED visit is exponential($\lambda = 1.8$)
- the time until the second asthma ED visit is gamma($r = 2$, $\lambda = 1.8$)
- the time until the tenth asthma ED visit is gamma($r = 10$, $\lambda = 1.8$).

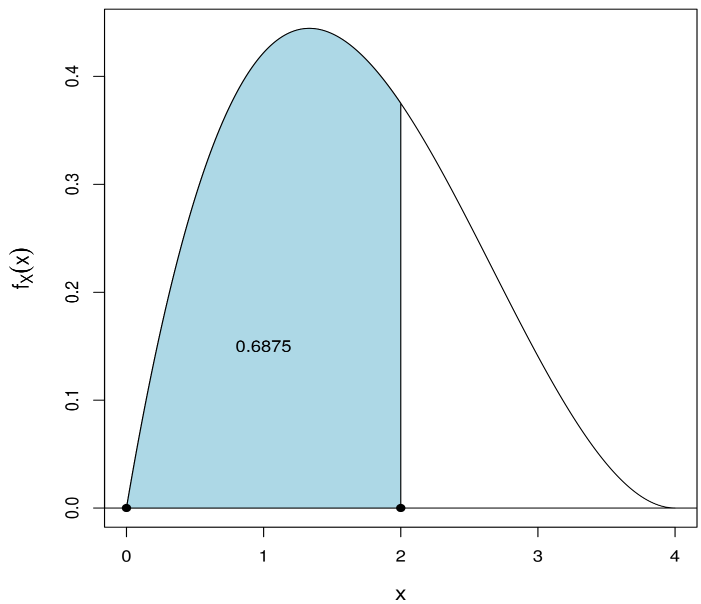
(c) Think of each day as a "trial" and having at least two asthma ED visits on a given day as a "success." Then Y counts the number of "successes" in $n = 31$ "trials." We know Y has a binomial distribution if the Bernoulli trial assumptions hold. These are:

1. each day either has at least two asthma ED visits or it doesn't
2. the number of ED visits each day are **independent** across the 31 days
3. the probability of having at least two asthma ED visits, $p \approx 0.537$, is the same each day.

5. (a) We want

$$P(X < 2) = \int_0^2 f_X(x)dx = \int_0^2 \frac{3x}{64}(4 - x)^2 dx.$$

This probability is shown as the shaded area under $f_X(x)$ at the top of the next page.



The last integral equals

$$\begin{aligned}\frac{3}{64}\int_0^2 x(4-x)^2dx &= \frac{3}{64}\int_0^2 x(16-8x+x^2)dx \\ &= \frac{3}{64}\int_0^2 (16x-8x^2+x^3)dx \\ &= \frac{3}{64}\left(\frac{16x^2}{2}-\frac{8x^3}{3}+\frac{x^4}{4}\right)\Bigg|_0^2 = \frac{3}{64}\left(32-\frac{64}{3}+4\right) = 0.6875.\end{aligned}$$

(b) The expected value (mean) of X is

$$\begin{aligned}E(X)=\int_0^4 xf_X(x)dx &= \frac{3}{64}\int_0^4 x^2(4-x)^2dx \\ &= \frac{3}{64}\int_0^4 x^2(16-8x+x^2)dx \\ &= \frac{3}{64}\int_0^2 (16x^2-8x^3+x^4)dx \\ &= \frac{3}{64}\left(\frac{16x^3}{3}-\frac{8x^4}{4}+\frac{x^5}{5}\right)\Bigg|_0^4 = \frac{3}{64}\left[\frac{16(64)}{3}-512+204.8\right] = 1.6.\end{aligned}$$

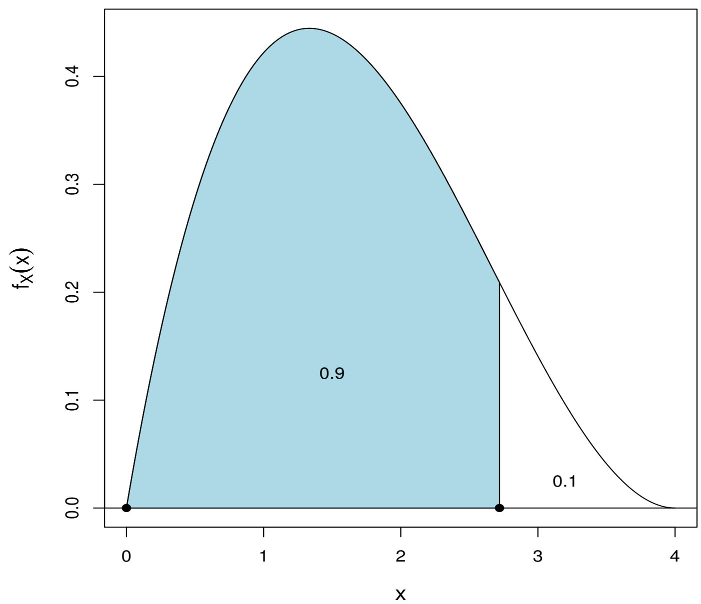
This is the "balance point" of the distribution above.
(c) The 90th percentile (0.9 quantile) $\phi_{0.9}$ is the value that solves

$$\int_0^{\phi_{0.9}} f_X(x)dx = \int_0^{\phi_{0.9}} \frac{3x}{64}(4-x)^2dx = 0.9.$$

That is, the area to the left of $\phi_{0.9}$ is 0.90. Equivalently, $\phi_{0.9}$ also solves

$$\int_{\phi_{0.9}}^4 f_X(x)dx = \int_{\phi_{0.9}}^4 \frac{3x}{64}(4-x)^2dx = 0.1.$$

That is, the area to the right of $\phi_{0.9}$ is 0.10. See the figure at the top of the next page.



6. (a) We want

$$P(T < 3) = F_T(3) = 1 - \exp\left[-\left(\frac{3}{5.5}\right)^{2.5}\right] \approx 0.197.$$

(b) The median $\phi_{0.5}$ solves

$$\begin{aligned}
P(T < \phi_{0.5}) = F_T(\phi_{0.5}) \quad &= \quad 1 - \exp\left[-\left(\frac{\phi_{0.5}}{5.5}\right)^{2.5}\right] \overset{\text{set}}{=} 0.5 \\
&\implies \exp\left[-\left(\frac{\phi_{0.5}}{5.5}\right)^{2.5}\right] = 0.5 \\
&\implies -\left(\frac{\phi_{0.5}}{5.5}\right)^{2.5} = \ln(0.5) \\
&\implies \left(\frac{\phi_{0.5}}{5.5}\right)^{2.5} = -\ln(0.5) \\
&\implies \frac{\phi_{0.5}}{5.5} = [-\ln(0.5)]^{1/2.5} \\
&\implies \phi_{0.5} = 5.5[-\ln(0.5)]^{1/2.5} \approx 4.75 \text{ years.}
\end{aligned}$$

(c) The hazard function $h_T(t)$ is increasing. This means the population of tubes is getting weaker over time.
(d) She could try using any distribution where the support is positive. The gamma (Chapter 4) and lognormal (HW4) are other options. There are many others too that we haven't talked about.