1. An electronic system has two types of components in operation. Let Y_1 and Y_2 denote the lifetimes of the type 1 and type 2 components, respectively. The joint probability density function (pdf) of Y_1 and Y_2 is

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{1}{8} y_1 e^{-(y_1+y_2)/2}, & y_1 > 0,\ y_2 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Calculate $P(Y_1 - Y_2 > 0)$.
(b) Find the expected value of the ratio; i.e.,

$$E\left(\frac{Y_2}{Y_1}\right).$$

(c) Define $U = Y_1 + 2Y_2$. Find the mean and variance of U.

2. A committee of 2 people is randomly selected from a group containing 3 Republicans, 2 Democrats, and 1 Libertarian. Let Y_1 denote the number of Republicans on the committee and let Y_2 denote the number of Democrats on the committee.
(a) Write out the joint probability mass function (pmf) of Y_1 and Y_2 using a two-way table.
(b) Calculate $E(Y_1|Y_2 = 1)$.
(c) Calculate ρ, the correlation of Y_1 and Y_2.

3. Suppose Y_1 and Y_2 have the joint probability density function (pdf)

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 6y_1^2 y_2, & 0 < y_1 < y_2,\ y_1 + y_2 \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Derive the marginal pdf of Y_1.
(b) Find $P(Y_2 < 1.1|Y_1 = 0.50)$.
(c) Are Y_1 and Y_2 independent? Explain.

4. When a current Y_1 (measured in amperes) flows through a resistance Y_2 (measured in ohms), the power generated is given $W = Y_1^2 Y_2$ (measured in watts). Suppose the marginal distributions of Y_1 and Y_2, respectively, are

$$f_{Y_1}(y_1) = \begin{cases} 6y_1(1 - y_1), & 0 < y_1 < 1, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_{Y_2}(y_2) = \begin{cases} 2y_2, & 0 < y_2 < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) The random variables Y_1 and Y_2 both have beta distributions. Give the parameters associated with each distribution.
(b) Assuming Y_1 and Y_2 are independent, compute $E(W)$. Could you find $E(W)$ without assuming Y_1 and Y_2 are independent? Explain.
(c) Find $E(Y_1^2 + Y_2^2)$. Assuming Y_1 and Y_2 are independent, find $V(Y_1^2 + Y_2^2)$.

5. The continuous random vector $\mathbf{Y} = (Y_1, Y_2)$ has probability density function (pdf)

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{6}{7}\left(y_1^2 + \frac{y_1 y_2}{2}\right), & 0 < y_1 < 1,\ 0 < y_2 < 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Compute $P(Y_1 < 2Y_2)$.
(b) Find $f_{Y_2|Y_1}(y_2|y_1)$, the conditional pdf of Y_2 given $Y_1 = y_1$, and calculate $E(Y_2|Y_1 = y_1)$.
(c) Compute $\text{Cov}(Y_1, Y_2)$.

6. An engineering system consists of two components operating independently of each other. Let Y_1 denote the time until component 1 fails, and let Y_2 denote the time until component 2 fails. An engineer models Y_1 as an exponential random variable with mean 1, and Y_2 as a gamma random variable with $\alpha = \beta = 2$.
(a) Find the joint probability density function (pdf) of Y_1 and Y_2. Note the support.
(b) Find the probability component 1 fails after component 2 does; i.e., find $P(Y_1 > Y_2)$.

7. Suppose that Y_1, Y_2, and Y_3 are random variables with

$$\begin{array}{ccc} E(Y_1) = 1 & E(Y_2) = 2 & E(Y_3) = 3 \\ V(Y_1) = 1 & V(Y_2) = 4 & V(Y_3) = 9 \\ \text{Cov}(Y_1, Y_2) = 0 & \text{Cov}(Y_1, Y_3) = 1 & \text{Cov}(Y_2, Y_3) = -1. \end{array}$$

Define the linear combination $U = 3Y_1 - 2Y_2 + 6Y_3$. Find the mean and variance of U.

8. A large insurance agency services a number of customers who have purchased both a homeowner's policy and an automobile policy from the agency. For each type of policy a deductible must be specified. For the homeowner policy, the choices are \$100, \$250, and \$500. For the automobile policy, the choices are \$0, \$100, \$250, and \$500. Let Y_1 and Y_2 denote the homeowner policy deductible and automobile policy deductible, respectively. Actuaries have provided us with the joint probability mass function (pmf) of Y_1 and Y_2 in the table below.

	$y_2 = 0$	$y_2 = 100$	$y_2 = 250$	$y_2 = 500$
$y_1 = 100$	0.02	0.10	0.10	0.08
$y_1 = 250$	0.12	0.12	0.10	0.06
$y_1 = 500$	0.06	0.08	0.10	0.06

(a) Find the marginal pmf of Y_1.
(b) Find the marginal cumulative distribution function (cdf) of Y_1.
(c) Find the conditional pmf of Y_2, given $Y_1 = 250$.
(d) Find the conditional mean and variance of Y_2, given $Y_1 = 250$.

9. Suppose the random vector $\mathbf{Y} = (Y_1, Y_2)$ has probability density function (pdf)

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} c, & 0 < y_1 < 2,\ 0 < y_2 < 1,\ 2y_2 < y_1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch the support region of $\mathbf{Y} = (Y_1, Y_2)$ in the (y_1, y_2) plane. Place y_1 on the horizontal axis and y_2 on the vertical axis.
(b) Find the value of c that makes $f_{Y_1,Y_2}(y_1, y_2)$ a valid pdf.

(c) Describe, in words, what the function $f_{Y_1,Y_2}(y_1, y_2)$ looks like.
(d) Compute $\text{Cov}(Y_1, Y_2)$. Are Y_1 and Y_2 independent?

10. An electronic device is designed to switch house lights on and off at random times after it has been activated. Assume that the device is designed in such a way that it will be switched on and off exactly once in a one-hour period. Let Y_1 denote the time (in hours) at which the lights are turned on, and let Y_2 denote the time (in hours) at which the lights are turned off. The joint probability density function (pdf) for Y_1 and Y_2 is given by

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 8y_1y_2, & 0 < y_1 < y_2 < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Are Y_1 and Y_2 are independent? Explain.
(b) What is the probability that the device will turn on and turn off in less than 30 minutes after it has been activated?
(c) Find $E(Y_2|Y_1 = 0.5)$ and $V(Y_2|Y_1 = 0.5)$.

11. Suppose Y_1, Y_2, and Y_3 are random variables. Using the definition of covariance, prove that

$$\text{Cov}(Y_1, Y_2 + Y_3) = \text{Cov}(Y_1, Y_2) + \text{Cov}(Y_1, Y_3).$$

12. In a genetics model, the proportion, say Y_1, of a population with trait 1 is always less than the proportion, say Y_2, of a population with trait 2. Suppose (Y_1, Y_2) has joint probability density function (pdf)

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 6y_1, & 0 < y_1 < y_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) If subjects in the population possessing trait 1 always possess trait 2, then $Y_2 - Y_1$ denotes the proportion of the population which has trait 2, but not trait 1. Compute $E(Y_2 - Y_1)$.
(b) Find the conditional distribution of Y_2, given $Y_1 = y_1$.
(c) Find the mean and variance of $E(Y_2|Y_1)$.

13. Suppose X_1 and X_2 are independent random variables with $E(X_1) = E(X_2) = 0$, $V(X_1) = 1$, and $V(X_2) = 4$. Define

$$\begin{aligned} U_1 &= X_1 + X_2 \\ U_2 &= X_1 - X_2. \end{aligned}$$

Calculate ρ, the correlation of U_1 and U_2.

14. The management at a fast-food outlet is interested in the joint behavior of the random variables Y_1 and Y_2. The variable Y_1 denotes the total time between a customers arrival at the store and his/her departure from the service window. The variable Y_2 denotes the time a customer waits in line before reaching the service window. Both Y_1 and Y_2 are measured in

minutes. The joint distribution of Y_1 and Y_2 is

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 < y_2 < y_1 < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(a) The quantity $Y_1 - Y_2$ denotes the time spent at the service window. Compute $P(Y_1 - Y_2 > 1)$.
(b) Find $E(Y_1 - Y_2)$.
(c) If 2 minutes elapse between a customer's arrival at the store and his departure from the service window, find the probability that he waited in line less than 1 minute to reach the window. That is, compute $P(Y_2 < 1|Y_1 = 2)$.

15. An insurance company offers two types of earthquake insurance, Type I and Type II. Let Y_1 and Y_2 denote the claim amounts (in \$10,000s) for the two types, respectively, and assume that the joint pdf of Y_1 and Y_2 is

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{1}{125} y_1 e^{-(y_1+y_2)/5}, & y_1 > 0,\ y_2 > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(b) Find the probability that the next claim amount for Type I is larger than twice the next claim amount for Type II. That is, find $P(Y_1 > 2Y_2)$.
(c) Calculate the variance of $U = 2Y_1 - Y_2$.

16. For a nationally administered aptitude exam, let X denote a subject's verbal score and let Y denote the subject's quantitative score. Scores have been standardized to fall between 0 and 1; in particular, the joint probability density function (pdf) of (X, Y) is

$$f_{X,Y}(x, y) = \begin{cases} \frac{2}{5}(2x + 3y), & 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Derive the marginal pdf of Y.
(b) Find the conditional pdf of Y given $X = x$. Are X and Y independent? Explain.

17. Suppose the random vector $\mathbf{Y} = (Y_1, Y_2)$ has probability density function (pdf)

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \dfrac{e^{-y_1/y_2} e^{-y_2}}{y_2}, & y_1 > 0,\ y_2 > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $E(Y_1|Y_2 = y_2)$.
(b) Find the correlation of Y_1 and Y_2.

18. Suppose the random vector $\mathbf{Y} = (Y_1, Y_2)$ has probability density function (pdf)

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} y_1 e^{-y_1(1+y_2)}, & y_1 > 0,\ y_2 > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show $E(Y_2)$ does not exist but that $E(Y_2|Y_1 = y_1) = 1/y_1$.
(b) Find the variance of $E(Y_2|Y_1)$.