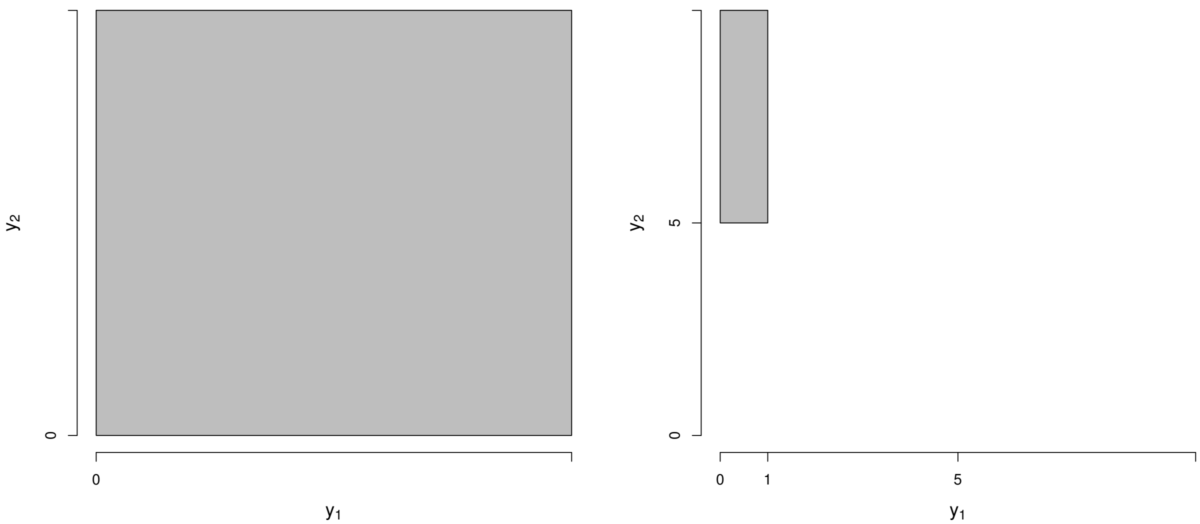
5.7. The support is $R = \{(y_1, y_2) : y_1 > 0, y_2 > 0\}$, the entire first quadrant. See the picture below (left):



The joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ is a three-dimensional function which takes the value $e^{-(y_1+y_2)}$ over this region (i.e., the entire first quadrant) and is otherwise equal to zero.

(a) We calculate $P(Y_1 < 1, Y_2 > 5)$ by integrating the joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ over the set

$$B = \{(y_1, y_2) : y_1 < 1, y_2 > 5\}.$$

This set is shown above (right). The limits come from the picture. Therefore,

$$\begin{aligned}
P(Y_1 < 1, Y_2 > 5) = \int_{y_1=0}^{1} \int_{y_2=5}^{\infty} e^{-(y_1+y_2)}\, dy_2 dy_1 &= \int_{y_1=0}^{1} e^{-y_1} \left(\int_{y_2=5}^{\infty} e^{-y_2}\, dy_2 \right) dy_1 \\
&= \int_{y_1=0}^{1} e^{-y_1} \left[\left(-e^{-y_2} \Big|_{y_2=5}^{\infty} \right) \right] dy_1 \\
&= \int_{y_1=0}^{1} e^{-y_1} (0 + e^{-5}) dy_1 \\
&= e^{-5} \int_{y_1=0}^{1} e^{-y_1} dy_1 \\
&= e^{-5} \left(-e^{-y_1} \Big|_{y_1=0}^{1} \right) = e^{-5}(1 - e^{-1}) \approx 0.004.
\end{aligned}$$

This is the (approximate) volume under $f_{Y_1,Y_2}(y_1, y_2)$ over the set B above.
(b) We calculate $P(Y_1 + Y_2 < 3)$ by integrating the joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ over the set

$$B = \{(y_1, y_2) : 0 < y_1 + y_2 < 3\}.$$

This set is shown on the next page (top). Note that the boundary of this set is

$$y_1 + y_2 = 3 \implies y_2 = 3 - y_1,$$

a linear function of y_1 with slope -1 and intercept 3.

The limits come from the picture. Therefore,

$$
\begin{aligned}
P(Y_1+Y_2<3) = \int_{y_1=0}^{3}\int_{y_2=0}^{3-y_1} e^{-(y_1+y_2)}\ dy_2dy_1 &= \int_{y_1=0}^{3} e^{-y_1}\left(\int_{y_2=0}^{3-y_1} e^{-y_2}\ dy_2\right)dy_1 \\
&= \int_{y_1=0}^{3} e^{-y_1}\left[\left(-e^{-y_2}\Big|_{y_2=0}^{3-y_1}\right)\right]dy_1 \\
&= \int_{y_1=0}^{3} e^{-y_1}[1-e^{-(3-y_1)}]dy_1 \\
&= \int_{y_1=0}^{3} (e^{-y_1}-e^{-3})dy_1 \\
&= \left(-e^{-y_1}-e^{-3}y_1\right)\Big|_{y_1=0}^{3} \\
&= -\left[\left(e^{-3}+3e^{-3}\right)-(1+0)\right] = 1-4e^{-3} \approx 0.801.
\end{aligned}
$$

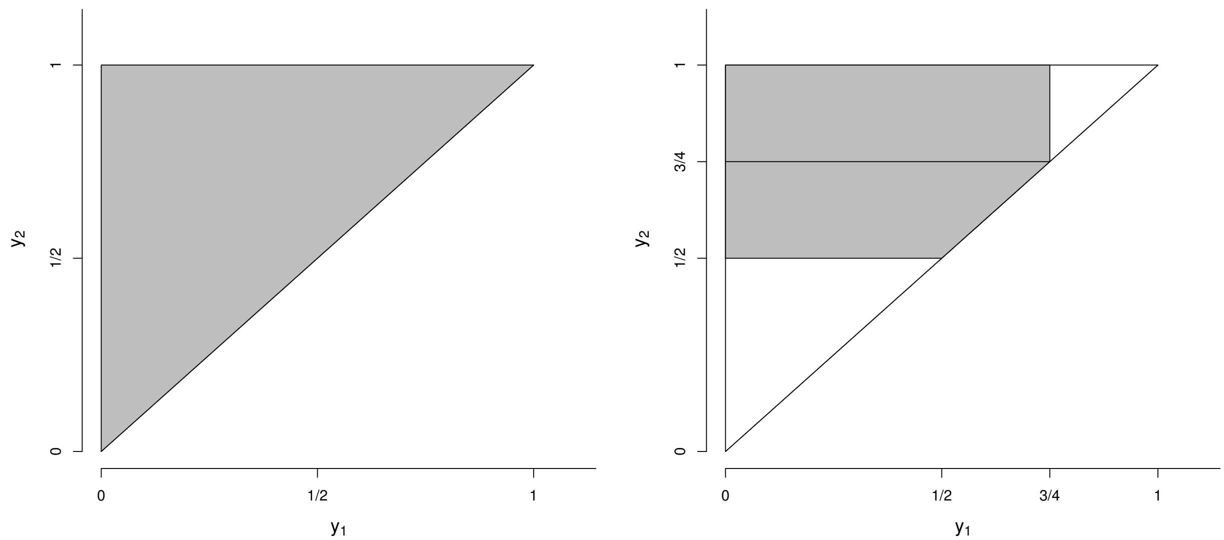
This is the (approximate) volume under $f_{Y_1,Y_2}(y_1,y_2)$ over the set B above.

5.9. The support $R=\{(y_1,y_2): 0\le y_1\le y_2\le 1\}$ is a triangular region. See the picture on the next page (left). The joint pdf $f_{Y_1,Y_2}(y_1,y_2)$ is a three-dimensional function which takes the value $k(1-y_2)$ over this region and is otherwise equal to zero.

(a) We know the joint pdf $f_{Y_1,Y_2}(y_1,y_2)$ integrates to 1 over the support. The limits come from the picture. Therefore,

$$
\begin{aligned}
1 \stackrel{\text{set}}{=} \int_{y_2=0}^{1}\int_{y_1=0}^{y_2} k(1-y_2)\ dy_1dy_2 &= k\int_{y_2=0}^{1}(1-y_2)\left(y_1\Big|_{y_1=0}^{y_2}\right)dy_2 \\
&= k\int_{y_2=0}^{1} y_2(1-y_2)dy_2 = k\frac{\Gamma(2)\Gamma(2)}{\Gamma(4)} = \frac{k}{6} \implies k=6.
\end{aligned}
$$

Note that the integrand in $\int_{y_2=0}^{1} y_2(1-y_2)dy_2$ is the kernel of the beta$(2,2)$ pdf and we are integrating over $(0,1)$. Therefore, this integral can be done quickly.



(b) The joint pdf of Y_1 and Y_2 is

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} 6(1-y_2), & 0 \le y_1 \le y_2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

We calculate $P(Y_1 \le 3/4, Y_2 \ge 1/2)$ by integrating the joint pdf $f_{Y_1,Y_2}(y_1,y_2)$ over the set

$$B = \{(y_1,y_2) : y_1 \le 3/4,\ y_2 \ge 1/2\}.$$

This set is shown above (right). The limits come from the picture. We have to break this up into two integrals:

$$\int_{y_2=3/4}^{1}\int_{y_1=0}^{3/4} 6(1-y_2)\ dy_1 dy_2 \quad \longleftarrow \quad \text{over upper rectangular region}$$

and

$$\int_{y_2=1/2}^{3/4}\int_{y_1=0}^{y_2} 6(1-y_2)\ dy_1 dy_2 \quad \longleftarrow \quad \text{over lower trapezoidal region}$$

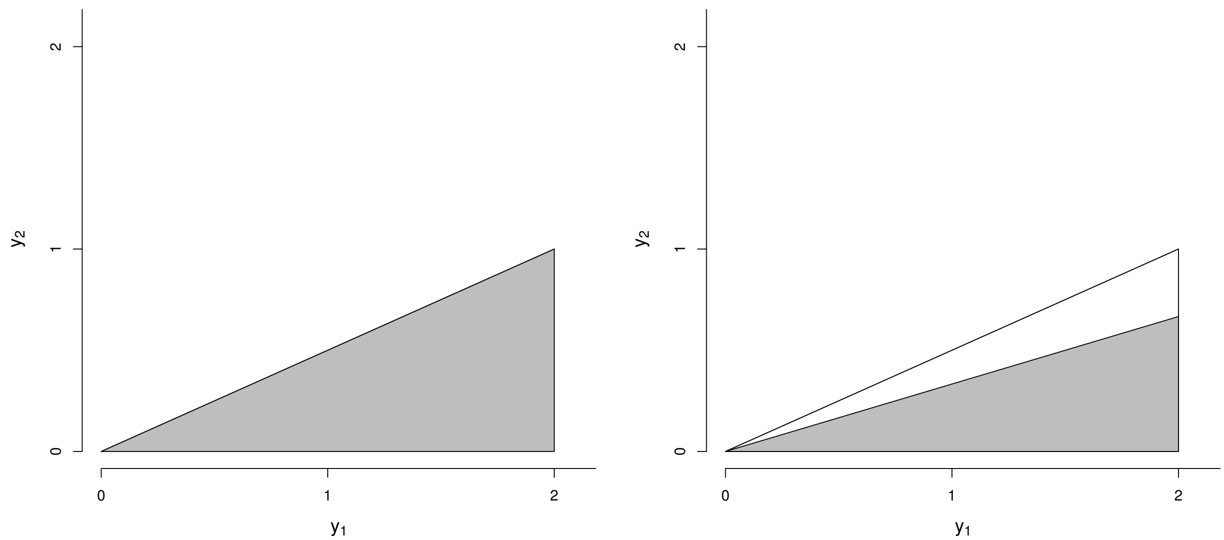
We do both integrals and then add them. The first integral is

$$\int_{y_2=3/4}^{1} 6(1-y_2)\left(\frac{3}{4}\right) dy_2 = 6\left(\frac{3}{4}\right)\left(y_2 - \frac{y_2^2}{2}\right)\Bigg|_{y_2=3/4}^{1} = 6\left(\frac{3}{4}\right)\left(1 - \frac{1}{2} - \frac{3}{4} + \frac{9}{32}\right) \approx 0.141.$$

The second integral is

$$\begin{aligned}\int_{y_2=1/2}^{3/4}\int_{y_1=0}^{y_2} 6(1-y_2)\ dy_1 dy_2 = \int_{y_2=1/2}^{3/4} 6y_2(1-y_2)\ dy_1 dy_2 &= 6\left(\frac{y_2^2}{2} - \frac{y_2^3}{3}\right)\Bigg|_{y_2=1/2}^{3/4} \\ &= 6\left(\frac{9}{32} - \frac{27}{192} - \frac{1}{8} + \frac{1}{24}\right) \approx 0.344.\end{aligned}$$

Therefore, $P(Y_1 \le 3/4, Y_2 \ge 1/2) \approx 0.141 + 0.344 = 0.485$. This is the (approximate) volume under $f_{Y_1,Y_2}(y_1,y_2)$ over the set B above.



5.10. The support

$$R = \{(y_1, y_2) : 0 \le y_1 \le 2, 0 \le y_2 \le 1, y_1 \ge 2y_2\}$$

is the triangular region in the picture above (left). Note that the boundary of this support is

$$y_1 = 2y_2 \implies y_2 = \frac{y_1}{2},$$

a linear function of y_1 with slope 1/2 and intercept 0. The joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ is a three-dimensional function which takes the value k over this region and is otherwise equal to zero. In other words, the joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ is **constant** (with height k) over this triangle.

(a) We know the joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ integrates to 1 over the support. We could write

$$1 \overset{\text{set}}{=} \int_{y_1=0}^{2} \int_{y_2=0}^{y_1/2} k \; dy_2 dy_1$$

and solve for k. However, we could also find k from using elementary geometry. The area of the triangle (base/support) is

$$\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 1 \times 1 = 1.$$

Therefore, the height of the function $f_{Y_1,Y_2}(y_1, y_2)$ must be 1 in order for the volume under $f_{Y_1,Y_2}(y_1, y_2)$ to equal 1. Therefore, $k = 1$. If you solve the integral equation above for k, you will also get $k = 1$.

(b) We calculate $P(Y_1 \ge 3Y_2)$ by integrating the joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ over the set

$$B = \{(y_1, y_2) : y_1 \ge 3y_2\}.$$

This set is shown above (right). Note that the boundary of B is

$$y_1 = 3y_2 \implies y_2 = \frac{y_1}{3},$$

a linear function of y_1 with slope 1/3 and intercept 0. The limits come right from this picture; i.e., we could calculate

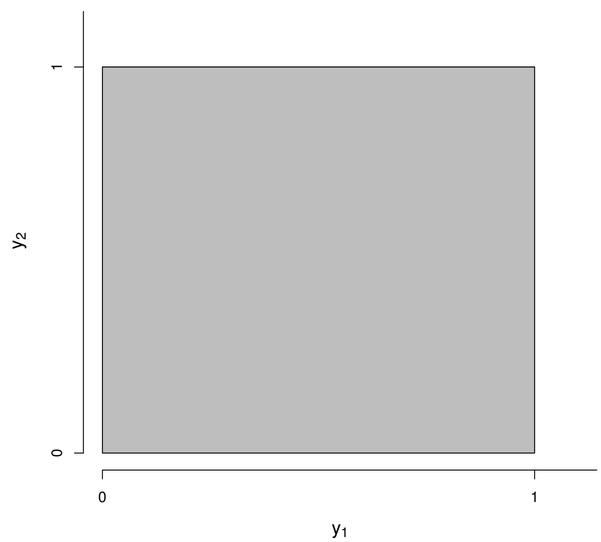
$$P(Y_1 \geq 3Y_2) = \int_{y_1=0}^{2} \int_{y_2=0}^{y_1/3} 1 \; dy_2 dy_1.$$

However, again because the height of the joint pdf $f_{Y_1,Y_2}(y_1, y_2) = 1$ (constant), we can calculate this volume quickly using geometry again:

$$\text{volume} \;=\; \underbrace{\frac{1}{2} \times \text{base} \times \text{height}}_{\text{area of } B} \;\times\; \text{height of pdf} \;=\; \frac{1}{2} \times 2 \times \frac{2}{3} \times 1 = \frac{2}{3}.$$

You would get the same answer if you did the double integral above.

5.16. The support is $R = \{(y_1, y_2) : 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1\}$, the unit square. See below:



The joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ is a three-dimensional function which takes the value $y_1 + y_2$ over this region and is otherwise equal to zero.

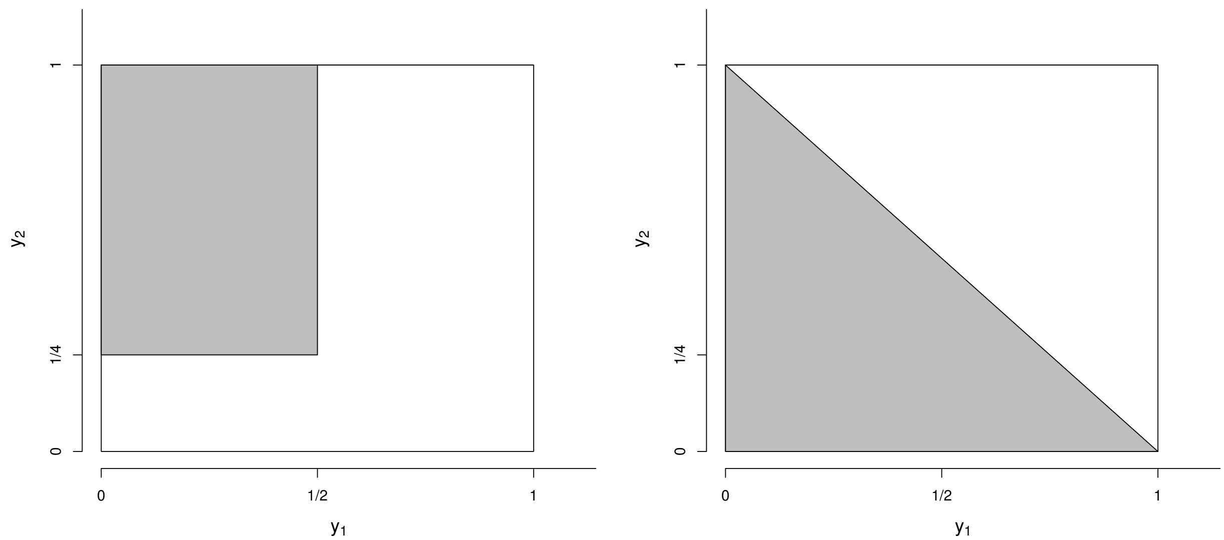
(a) We calculate $P(Y_1 < 1/2, Y_2 > 1/4)$ by integrating the joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ over the set

$$B = \{(y_1, y_2) : y_1 < 1/2,\; y_2 > 1/4\}.$$

This set is shown on the next page (left). The limits come from the picture. Therefore,

$$\begin{aligned}
P(Y_1 < 1/2, Y_2 > 1/4) &= \int_{y_2=1/4}^{1} \int_{y_1=0}^{1/2} (y_1 + y_2) \; dy_1 dy_2 \\
&= \int_{y_2=1/4}^{1} \left(\frac{y_1^2}{2} + y_1 y_2 \right) \Bigg|_{y_1=0}^{1/2} dy_2 \\
&= \int_{y_2=1/4}^{1} \left(\frac{1}{8} + \frac{y_2}{2} \right) dy_2 \\
&= \left(\frac{y_2}{8} + \frac{y_2^2}{4} \right) \Bigg|_{y_2=1/4}^{1} = \frac{1}{8} + \frac{1}{4} - \frac{1}{32} - \frac{1}{64} = \frac{21}{64} \approx 0.328.
\end{aligned}$$

This is the volume under $f_{Y_1,Y_2}(y_1, y_2)$ over the set B above (picture on next page).



(b) We calculate $P(Y_1 + Y_2 \leq 1)$ by integrating the joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ over the set

$$B = \{(y_1, y_2) : y_1 + y_2 \leq 1\}.$$

This set is shown above (right). Note that the boundary of B is

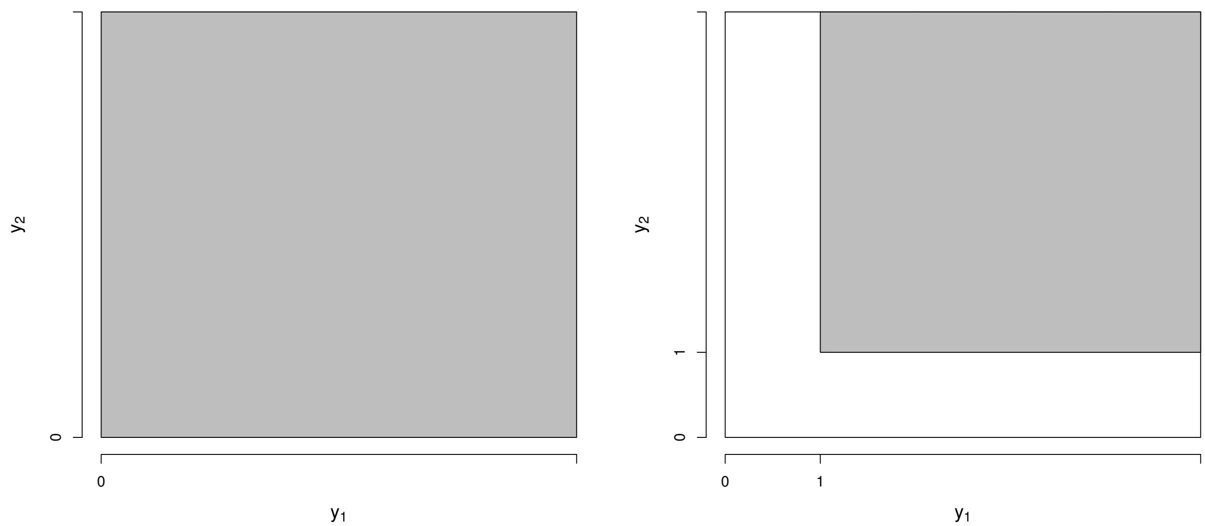
$$y_1 + y_2 = 1 \implies y_2 = 1 - y_1,$$

a linear function of y_1 with slope -1 and intercept 1. Therefore,

$$\begin{aligned}
P(Y_1 + Y_2 \leq 1) &= \int_{y_1=0}^{1} \int_{y_2=0}^{1-y_1} (y_1 + y_2)\ dy_1 dy_2 \\
&= \int_{y_1=0}^{1} \left(y_1 y_2 + \frac{y_2^2}{2} \right) \Big|_{y_2=0}^{1-y_1} dy_1 \\
&= \int_{y_1=0}^{1} \left[y_1(1 - y_1) + \frac{(1 - y_1)^2}{2} \right] dy_1 \\
&= \int_{y_1=0}^{1} y_1(1 - y_1) dy_1 + \frac{1}{2} \int_{y_1=0}^{1} (1 - y_1)^2 dy_1 \\
&= \frac{\Gamma(2)\Gamma(2)}{\Gamma(4)} + \frac{1}{2} \frac{\Gamma(1)\Gamma(3)}{\Gamma(4)} = \frac{1}{6} + \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{3}.
\end{aligned}$$

This is the volume under $f_{Y_1,Y_2}(y_1, y_2)$ over the set B above. Note that the integrand in the first integral is a beta$(2, 2)$ kernel. The integrand in the second integral is a beta$(1, 3)$ kernel. Both integrals are over $(0, 1)$. Therefore, these integrals can be done quickly.

5.18. The support is $R = \{(y_1, y_2) : y_1 > 0, y_2 > 0\}$, the entire first quadrant. See the picture on the next page (left). We calculate $P(Y_1 > 1, Y_2 > 1)$ by integrating the joint pdf $f_{Y_1,Y_2}(y_1, y_2)$



over the set

$$B = \{(y_1, y_2) : y_1 > 1,\ y_2 > 1\}.$$

This set is shown above (right). The limits come from the picture. Therefore,

$$\begin{aligned} P(Y_1 > 1, Y_2 > 1) &= \int_{y_2=1}^{\infty}\int_{y_1=1}^{\infty} \frac{1}{8} y_1 e^{-(y_1+y_2)/2}\ dy_1 dy_2 \\ &= \frac{1}{8}\int_{y_2=1}^{\infty} e^{-y_2/2}\left(\int_{y_1=1}^{\infty} y_1 e^{-y_1/2}\ dy_1\right) dy_2. \end{aligned}$$

Let's do the inner integral by parts:

$$\begin{aligned} u &= y_1 & du &= dy_1 \\ dv &= e^{-y_1/2} & v &= -2e^{-y_1/2}. \end{aligned}$$

Therefore, the inner integral is

$$\begin{aligned} \int_{y_1=1}^{\infty} y_1 e^{-y_1/2}\ dy_1 &= -2y_1 e^{-y_1/2}\Big|_{y_1=1}^{\infty} - \int_{y_1=1}^{\infty} -2e^{-y_1/2}\ dy_1 \\ &= \left(0 + 2e^{-1/2}\right) + 2\left(-2e^{-y_1/2}\right)\Big|_{y_1=1}^{\infty} \\ &= 2e^{-1/2} + 2\left(2e^{-1/2} - 0\right) = 6e^{-1/2}. \end{aligned}$$

Finally,

$$\begin{aligned} P(Y_1 > 1, Y_2 > 1) &= \frac{6e^{-1/2}}{8}\int_{y_2=1}^{\infty} e^{-y_2/2} dy_2 \\ &= \frac{6e^{-1/2}}{8}\left(-2e^{-y_2/2}\right)\Big|_{y_2=1}^{\infty} = \frac{6e^{-1/2}}{8}\left(2e^{-1/2} - 0\right) = \frac{12e^{-1}}{8} \approx 0.552. \end{aligned}$$

This is the volume under $f_{Y_1,Y_2}(y_1, y_2)$ over the set B above.

5.22. The table below shows the joint pmf of Y_1 and Y_2.

$p_{Y_1,Y_2}(y_1,y_2)$	$y_1=0$	$y_1=1$	$p_{Y_2}(y_2)$
$y_2=0$	0.38	0.17	0.55
$y_2=1$	0.14	0.02	0.16
$y_2=2$	0.24	0.05	0.29
$p_{Y_1}(y_1)$	0.76	0.24	

(a) The marginal pmfs are in the margins; i.e., the marginal pmf of Y_1 is

y_1	0	1
$p_{Y_1}(y_1)$	0.76	0.24

Note that $Y_1 \sim \text{Bernoulli}(p = 0.24)$. The marginal pmf of Y_2 is

y_2	0	1	2
$p_{Y_2}(y_2)$	0.55	0.16	0.29

(b) The conditional pmf $p_{Y_2|Y_1}(y_2|y_1 = 0)$ describes the distribution of Y_2 when $Y_1 = 0$. This is a univariate pmf with three possible values of Y_2, namely, 0, 1, and 2. These conditional probabilities are calculated below:

$$\begin{aligned}
p_{Y_2|Y_1}(y_2=0|y_1=0) &= \frac{p_{Y_1,Y_2}(0,0)}{p_{Y_1}(0)} = \frac{0.38}{0.76} = 0.50 \\
p_{Y_2|Y_1}(y_2=1|y_1=0) &= \frac{p_{Y_1,Y_2}(0,1)}{p_{Y_1}(0)} = \frac{0.14}{0.76} \approx 0.184 \\
p_{Y_2|Y_1}(y_2=2|y_1=0) &= \frac{p_{Y_1,Y_2}(0,2)}{p_{Y_1}(0)} = \frac{0.24}{0.76} \approx 0.316.
\end{aligned}$$

We can display the conditional pmf of Y_2 given $Y_1 = 0$ in the following table:

y_2	0	1	2
$p_{Y_2|Y_1}(y_2|y_1=0)$	0.50	0.184	0.316

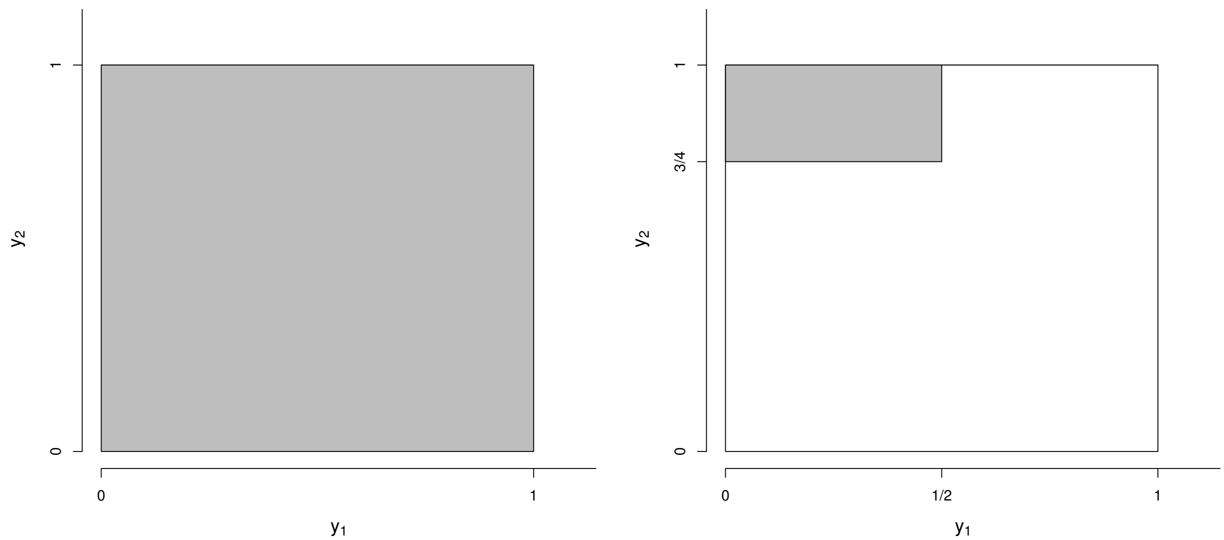
(c) We want $P(Y_1 = 0|Y_2 = 2)$. We can calculate this as follows:

$$P(Y_1=0|Y_2=2) = \frac{P(Y_1=0,Y_2=2)}{P(Y_2=2)} = \frac{p_{Y_1,Y_2}(0,2)}{p_{Y_2}(2)} = \frac{0.24}{0.29} \approx 0.828.$$

5.26. The support is $R = \{(y_1, y_2) : 0 \le y_1 \le 1, 0 \le y_2 \le 1\}$, the unit square; see next page (left). The joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ is a three-dimensional function which takes the value $4y_1y_2$ over this region and is otherwise equal to zero.

(a) The marginal pdf of Y_1 is nonzero when $0 \le y_1 \le 1$. For these values, the pdf is

$$f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{Y_1,Y_2}(y_1,y_2)\,dy_2 = \int_{y_2=0}^{1} 4y_1y_2 dy_2 = 4y_1\left(\frac{y_2^2}{2}\right)\Bigg|_{y_2=0}^{1} = 2y_1.$$



Therefore, the marginal pdf of Y_1 is given by

$$f_{Y_1}(y_1) = \begin{cases} 2y_1, & 0 \leq y_1 \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

You should recognize this as a beta pdf with parameters $\alpha = 2$ and $\beta = 1$. That is, marginally, $Y_1 \sim \text{beta}(2, 1)$. The exact same argument shows that $Y_2 \sim \text{beta}(2, 1)$; i.e., the pdf of Y_2 is

$$f_{Y_2}(y_2) = \begin{cases} 2y_2, & 0 \leq y_2 \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Note: You should see that Y_1 and Y_2 are independent because

$$f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2).$$

However, the notion of independence has not been introduced so far at this point in the textbook, so we will carry out all future calculations as is.

(b) This is a conditional probability:

$$P(Y_1 \leq 1/2 | Y_2 \geq 3/4) = \frac{P(Y_1 \leq 1/2, Y_2 \geq 3/4)}{P(Y_2 \geq 3/4)}.$$

We calculate the numerator probability $P(Y_1 \leq 1/2, Y_2 \geq 3/4)$ using the joint pdf of Y_1 and Y_2 and integrating it over the set $B = \{(y_1, y_2) : y_1 \leq 1/2, y_2 \geq 3/4\}$; see above (right):

$$\begin{aligned} P(Y_1 \leq 1/2, Y_2 \geq 3/4) &= \int_{y_1=0}^{1/2} \int_{y_2=3/4}^{1} 4y_1 y_2 \; dy_2 dy_1 \\ &= 4 \int_{y_1=0}^{1/2} y_1 \left(\frac{y_2^2}{2} \right) \Bigg|_{y_2=3/4}^{1} dy_1 = 4 \left(\frac{1}{2} - \frac{9}{32} \right) \left(\frac{y_1^2}{2} \right) \Bigg|_{y_1=0}^{1/2} = 4 \left(\frac{7}{32} \right) \frac{1}{8} = \frac{7}{64}. \end{aligned}$$

We can calculate the denominator probability $P(Y_2 \geq 3/4)$ using either the marginal pdf of Y_2 or the joint pdf of Y_1 and Y_2. Let's use the marginal because we already have it:

$$P(Y_2 \geq 3/4) = \int_{y_2=3/4}^{1} 2y_2 \; dy_2 = y_2^2 \Big|_{y_2=3/4}^{1} = 1 - \frac{9}{16} = \frac{7}{16}.$$

Therefore,

$$P(Y_1 \leq 1/2 | Y_2 \geq 3/4) = \frac{7/64}{7/16} = \frac{1}{4}.$$

Note: Because Y_1 and Y_2 are independent, note that

$$P(Y_1 \leq 1/2 | Y_2 \geq 3/4) = P(Y_1 \leq 1/2) = \int_{y_1=0}^{1/2} 2y_1 dy_1 = y_1^2 \Big|_{y_1=0}^{1/2} = \frac{1}{4}.$$

Independence makes our lives easier!

(c) Whatever the value of $Y_2 = y_2$ is, the possible values of Y_1 are $0 \leq y_1 \leq 1$; see the bivariate support on the last page. For these values, the conditional pdf of Y_1 is

$$f_{Y_1|Y_2}(y_1|y_2) = \frac{f_{Y_1,Y_2}(y_1, y_2)}{f_{Y_2}(y_2)} = \frac{4y_1 y_2}{2y_2} = 2y_1.$$

Summarizing,

$$f_{Y_1|Y_2}(y_1|y_2) = \begin{cases} 2y_1, & 0 \leq y_1 \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Note: The conditional pdf $f_{Y_1|Y_2}(y_1|y_2)$ is the same as the marginal pdf

$$f_{Y_1}(y_1) = \begin{cases} 2y_1, & 0 \leq y_1 \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

This makes sense because Y_1 and Y_2 are independent.

(d) This is completely analogous to part (c). The same argument shows

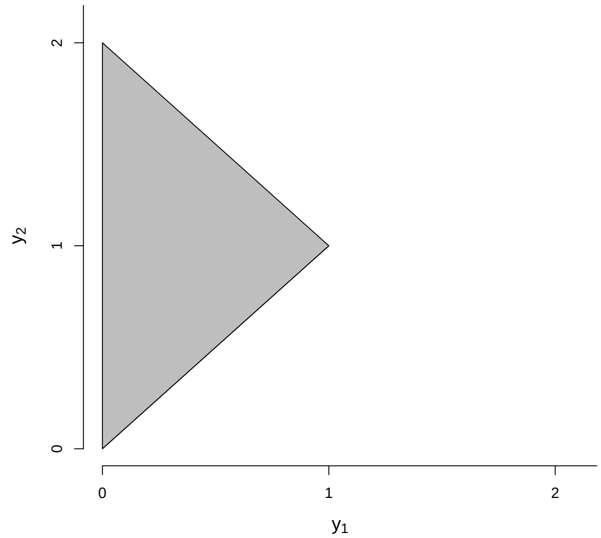
$$f_{Y_2|Y_1}(y_2|y_1) = \begin{cases} 2y_2, & 0 \leq y_2 \leq 1 \\ 0, & \text{otherwise,} \end{cases}$$

which is the same as the marginal pdf of Y_2.

(e) We calculate $P(Y_1 \leq 3/4 | Y_2 = 1/2)$ using the conditional pdf of Y_1 when $Y_2 = 1/2$; see the formula for $f_{Y_1|Y_2}(y_1|y_2)$ above. As we have already seen, the formula for $f_{Y_1|Y_2}(y_1|y_2)$ does not depend on y_2 (because of independence). We have

$$P(Y_1 \leq 3/4 | Y_2 = 1/2) = \int_{y_1=0}^{3/4} f_{Y_1|Y_2}(y_1|y_2 = 3/4) dy_1 = \int_{y_1=0}^{3/4} 2y_1 \; dy_1 = y_1^2 \Big|_{y_1=0}^{3/4} = \frac{9}{16}.$$

Note that this is the same answer you would have gotten had you just calculated $P(Y_1 \leq 3/4)$ using the marginal pdf of Y_1. This is true because Y_1 and Y_2 are independent.



5.32. This is the same pdf as Example 5.15 in the notes. The support is the triangular region $R = \{(y_1, y_2) : 0 \le y_1 \le y_2,\ y_1 + y_2 \le 2\}$; see above. The lower boundary line is $y_2 = y_1$ and the upper is $y_2 = 2 - y_1$. The joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ is a three-dimensional function which takes the value $6y_1^2 y_2$ over this region and is otherwise equal to zero.

(a) For $0 \le y_1 \le 1$, the marginal pdf of Y_1 is

$$\begin{aligned} f_{Y_1}(y_1) = \int_{y_2=y_1}^{2-y_1} 6y_1^2 y_2 \, dy_2 &= 6y_1^2 \left(\frac{y_2^2}{2}\right)\Bigg|_{y_2=y_1}^{2-y_1} \\ &= 3y_1^2 \left[(2-y_1)^2 - y_1^2\right] = 3y_1^2(4 - 4y_1 + y_1^2 - y_1^2) = 12y_1^2(1-y_1). \end{aligned}$$

Summarizing,

$$f_{Y_1}(y_1) = \begin{cases} 12y_1^2(1-y_1), & 0 \le y_1 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Note that $y_1^2(1-y_1)$ is the beta$(3,2)$ kernel and the support of Y_1 is over $(0,1)$. Therefore, $Y_1 \sim \text{beta}(3,2)$. It is easy to check that

$$12 = \frac{\Gamma(5)}{\Gamma(3)\Gamma(2)},$$

the constant out front.

(b) For $0 \le y_2 \le 2$, the marginal pdf of Y_2 depends on whether

- $0 \le y_2 \le 1$
- $1 < y_2 \le 2$;

see the support above (i.e., how we integrate over y_1 depends where y_2 is).

Case 1: If $0 \le y_2 \le 1$, then

$$f_{Y_2}(y_2) = \int_{y_1=0}^{y_2} 6y_1^2 y_2 \; dy_1 = 6y_2 \left(\frac{y_1^3}{3}\right)\Bigg|_{y_1=0}^{y_2} = 2y_2^4.$$

Case 2: If $1 < y_2 \le 2$, then

$$f_{Y_2}(y_2) = \int_{y_1=0}^{2-y_2} 6y_1^2 y_2 \; dy_1 = 6y_2 \left(\frac{y_1^3}{3}\right)\Bigg|_{y_1=0}^{2-y_2} = 2y_2(2-y_2)^3.$$

Summarizing,

$$f_{Y_2}(y_2) = \begin{cases} 2y_2^4, & 0 \le y_2 \le 1 \\ 2y_2(2-y_2)^3, & 1 < y_2 \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to show this is a valid density; i.e., it integrates to one.

(c) Recall that the conditional pdf of Y_2, given $Y_1 = y_1$, is given by

$$f_{Y_2|Y_1}(y_2|y_1) = \frac{f_{Y_1,Y_2}(y_1,y_2)}{f_{Y_1}(y_1)}.$$

Looking at the bivariate support (previous page), for a given value of y_1, the random variable Y_2 takes on values between y_1 (lower boundary line) and $2-y_1$ (upper boundary line). Therefore, the conditional $f_{Y_2|Y_1}(y_2|y_1) > 0$ when $y_1 \le y_2 \le 2-y_1$ and is otherwise equal to zero. We have

$$f_{Y_2|Y_1}(y_2|y_1) = \frac{6y_1^2 y_2}{12y_1^2(1-y_1)} = \left[\frac{1}{2(1-y_1)}\right] y_2,$$

a linear function of y_2 with intercept 0 and slope $1/[2(1-y_1)]$. Summarizing,

$$f_{Y_2|Y_1}(y_2|y_1) = \begin{cases} \left[\frac{1}{2(1-y_1)}\right] y_2, & y_1 \le y_2 \le 2-y_1 \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to show

$$\int_{y_2=y_1}^{2-y_1} \left[\frac{1}{2(1-y_1)}\right] y_2 \; dy_2 = 1;$$

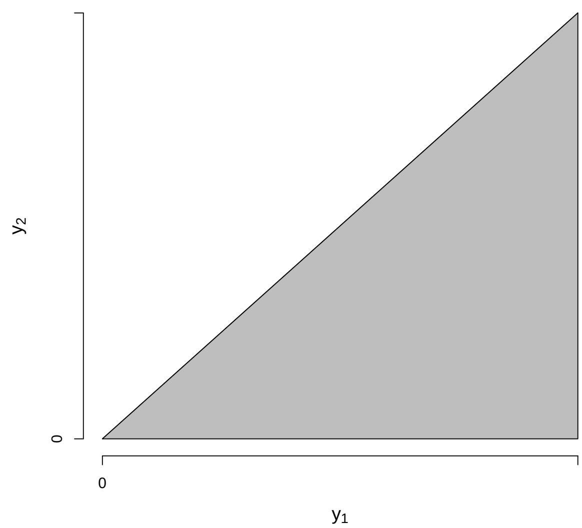
i.e., $f_{Y_2|Y_1}(y_2|y_1)$ is a valid density.

(d) When $Y_1 = 0.60$, the conditional pdf in part (c) becomes

$$f_{Y_2|Y_1}(y_2|y_1) = \begin{cases} \frac{5}{4}y_2, & 0.6 \le y_2 \le 1.4 \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$P(Y_2 < 1.1|Y_2 = 0.6) = \int_{0.6}^{1.1} \frac{5}{4}y_2 \; dy_2 = \frac{5}{4}\left(\frac{y_2^2}{2}\right)\Bigg|_{0.6}^{1.1} = \frac{5}{8}(1.1^2 - 0.6^2) \approx 0.531.$$



5.33. This is the same as Example 5.6 in the notes (except the roles of Y_1 and Y_2 are reversed). The support is the triangular region $R = \{(y_1, y_2) : 0 \leq y_2 \leq y_1\}$; see above. The upper boundary line is $y_2 = y_1$. The joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ is a three-dimensional function which takes the value e^{-y_1} over this region and is otherwise equal to zero.

(a) The marginal pdf of Y_1 is nonzero when $y_1 > 0$. For these values,

$$f_{Y_1}(y_1) = \int_{y_2=0}^{y_1} e^{-y_1} dy_2 = y_2 e^{-y_1}\Big|_{y_2=0}^{y_1} = y_1 e^{-y_1} - 0 = y_1 e^{-y_1}.$$

The marginal pdf of Y_2 is also nonzero when $y_2 > 0$. For these values,

$$f_{Y_2}(y_2) = \int_{y_1=y_2}^{\infty} e^{-y_1} dy_1 = -e^{-y_1}\Big|_{y_1=y_2}^{\infty} = -\left(\lim_{y_1\to\infty} e^{-y_1} - e^{-y_2}\right) = e^{-y_2}.$$

Summarizing, we have

$$f_{Y_1}(y_1) = \begin{cases} y_1 e^{-y_1}, & y_1 > 0, \\ 0, & \text{otherwise} \end{cases}$$

and

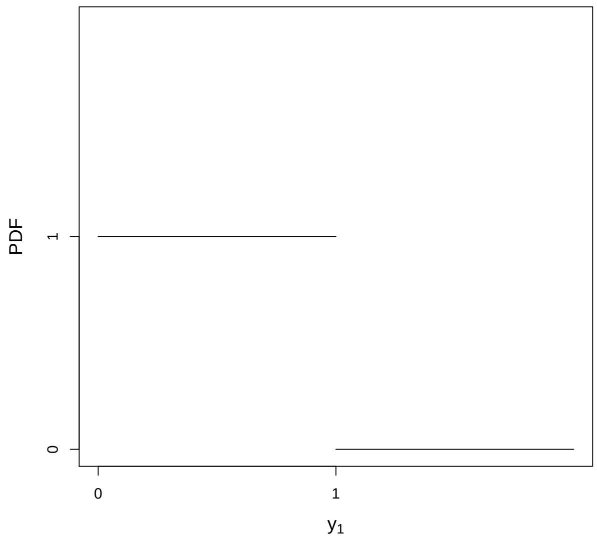
$$f_{Y_2}(y_2) = \begin{cases} e^{-y_2}, & y_2 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

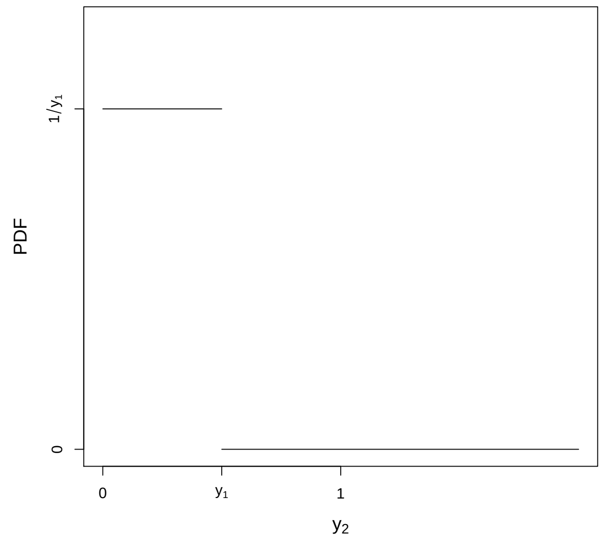
We recognize

$$\begin{aligned} Y_1 &\sim \text{gamma}(2, 1) \\ Y_2 &\sim \text{exponential}(1). \end{aligned}$$

(b) The conditional pdf of Y_1 is nonzero when $y_1 > y_2$, where $y_2 > 0$ is fixed. For these values,

$$f_{Y_1|Y_2}(y_1|y_2) = \frac{f_{Y_1,Y_2}(y_1, y_2)}{f_{Y_2}(y_2)} = \frac{e^{-y_1}}{e^{-y_2}} = e^{-(y_1-y_2)}.$$





Summarizing,

$$f_{Y_1|Y_2}(y_1|y_2) = \begin{cases} e^{-(y_1-y_2)}, & y_1 > y_2 \\ 0, & \text{otherwise.} \end{cases}$$

This is a shifted exponential(1) density where y_2 is the shift (to the right).

(c) The conditional pdf of Y_2 is nonzero when $0 < y_2 < y_1$, where y_1 is fixed. For these values,

$$f_{Y_2|Y_1}(y_2|y_1) = \frac{f_{Y_1,Y_2}(y_1,y_2)}{f_{Y_1}(y_1)} = \frac{e^{-y_1}}{y_1 e^{-y_1}} = \frac{1}{y_1}.$$

Summarizing,

$$f_{Y_2|Y_1}(y_2|y_1) = \begin{cases} \frac{1}{y_1}, & 0 < y_2 < y_1 \\ 0, & \text{otherwise} \end{cases}$$

Note that $Y_2|Y_1 = y_1 \sim \mathcal{U}(0, y_1)$.

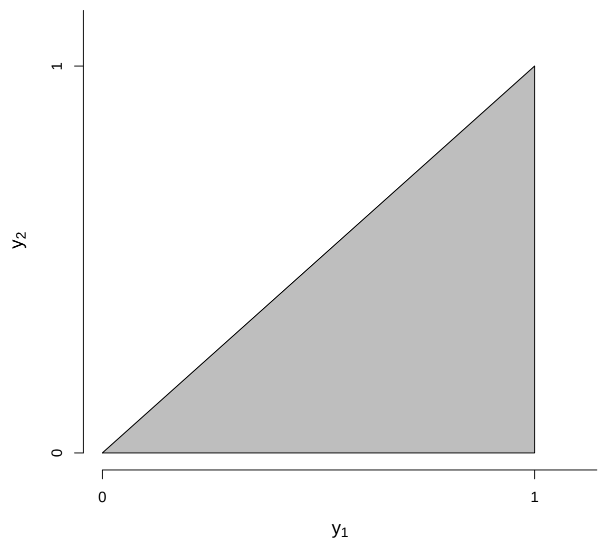
(d) It is easy to see that

$$f_{Y_1|Y_2}(y_1|y_2) \neq f_{Y_1}(y_1).$$

(e) The result in part (d) means that Y_1 and Y_2 are not independent. Therefore, the way probabilities are assigned to events involving Y_1 will be different, depending on the value of Y_2. In other words, $P(Y_1 \leq y_1|Y_2 = y_2)$ will not be the same as $P(Y_1 \leq y_1)$.

5.34. We are given that $Y_1 \sim \mathcal{U}(0, 1)$; i.e., the pdf of Y_1 is

$$f_{Y_1}(y_1) = \begin{cases} 1, & 0 < y_1 < 1 \\ 0, & \text{otherwise.} \end{cases}$$



This pdf is shown at the top of the last page (left). We are also given that the conditional distribution of $Y_2|Y_1 = y_1 \sim \mathcal{U}(0, y_1)$; i.e., the (conditional) pdf of Y_2 is

$$f_{Y_2|Y_1}(y_2|y_1) = \begin{cases} \dfrac{1}{y_1}, & 0 \le y_2 \le y_1 \\ 0, & \text{otherwise.} \end{cases}$$

This pdf is shown at the top of the last page (right).

(a) $Y_2|Y_1 = y_1 \sim \mathcal{U}(0, y_1)$
(b) We know that

$$f_{Y_2|Y_1}(y_2|y_1) = \frac{f_{Y_1,Y_2}(y_1, y_2)}{f_{Y_1}(y_1)} \implies f_{Y_1,Y_2}(y_1, y_2) = f_{Y_2|Y_1}(y_2|y_1) f_{Y_1}(y_1).$$

Therefore,

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \dfrac{1}{y_1}, & 0 \le y_2 \le y_1, 0 < y_1 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

The bivariate support of $\mathbf{Y} = (Y_1, Y_2)$ is $R = \{(y_1, y_2) : 0 \le y_2 \le y_1, 0 < y_1 < 1\}$; see above. The joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ is a three-dimensional function which takes the value $1/y_1$ over this region and is otherwise equal to zero.
(c) The marginal pdf of Y_2 is nonzero when $0 < y_2 < 1$. For these values,

$$f_{Y_2}(y_2) = \int_{y_1=y_2}^{1} \frac{1}{y_1} dy_1 = \ln y_1 \Big|_{y_1=y_2}^{1} = 0 - \ln y_2 = -\ln y_2.$$

Summarizing,

$$f_{Y_2}(y_2) = \begin{cases} -\ln y_2, & 0 < y_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to show this is a valid (marginal) density; i.e., it integrates to 1.