5.45. The joint pmf of

$$\begin{aligned} Y_1 &= \text{the number of contracts awarded to firm A} \\ Y_2 &= \text{the number of contracts awarded to firm B} \end{aligned}$$

is shown below, along with the marginal pmfs:

$p_{Y_1,Y_2}(y_1,y_2)$	$y_2 = 0$	$y_2 = 1$	$y_2 = 2$	$p_{Y_1}(y_1)$
$y_1 = 0$	1/9	2/9	1/9	4/9
$y_1 = 1$	2/9	2/9	0	4/9
$y_1 = 2$	1/9	0	0	1/9
$p_{Y_2}(y_2)$	4/9	4/9	1/9	

For Y_1 and Y_2 to be independent, we would need $p_{Y_1,Y_2}(y_1,y_2) = p_{Y_1}(y_1)p_{Y_2}(y_2)$ to hold for all (y_1,y_2) in the support

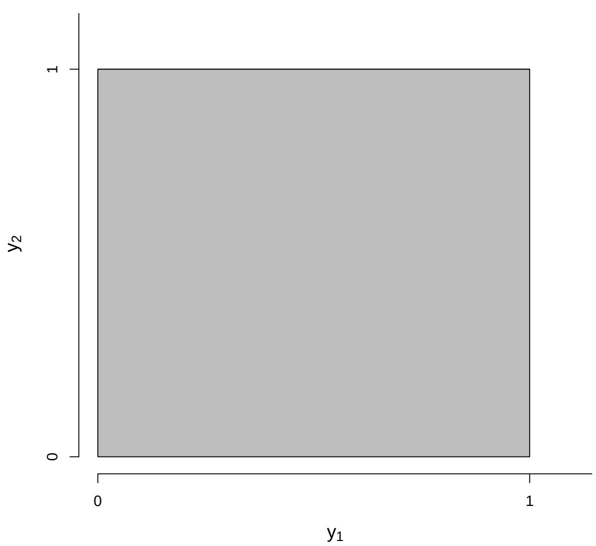
$$R = \{(0,0),(1,0),(2,0),(0,1),(1,1),(2,1),(0,2),(1,2),(2,2)\}.$$

However, this condition does not even hold for the first value $(0,0)$; i.e.,

$$\frac{1}{9} = p_{Y_1,Y_2}(0,0) \neq p_{Y_1}(0)p_{Y_2}(0) = \frac{4}{9}\left(\frac{4}{9}\right) = \frac{16}{81}.$$

Therefore, Y_1 and Y_2 are not independent.

5.52. We can quickly determine Y_1 and Y_2 are independent. First note that the support $R = \{(y_1,y_2) : 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1\}$, shown below, does not involve a constraint between y_1 and y_2.



Also, we can write

$$f_{Y_1,Y_2}(y_1,y_2) = 4y_1y_2 = 4y_1 \times y_2 = g(y_1)h(y_2),$$

where $g(y_1) = 4y_1$ and $h(y_2) = y_2$. Therefore, Y_1 and Y_2 are independent.

Note: The functions $g(y_1) = 4y_1$ and $h(y_2) = y_2$ are not the marginal pdfs, but they are proportional to them. For $0 < y_1 < 1$, the marginal pdf of Y_1 is

$$f_{Y_1}(y_1) = \int_{y_2=0}^{1} 4y_1y_2 \; dy_2 = 4y_1 \left(\frac{y_2^2}{2} \right) \Bigg|_{y_2=0}^{1} = 2y_1.$$

For $0 < y_2 < 1$, the marginal pdf of Y_2 is

$$f_{Y_2}(y_2) = \int_{y_1=0}^{1} 4y_1y_2 \; dy_1 = 4y_2 \left(\frac{y_1^2}{2} \right) \Bigg|_{y_1=0}^{1} = 2y_2.$$

Summarizing,

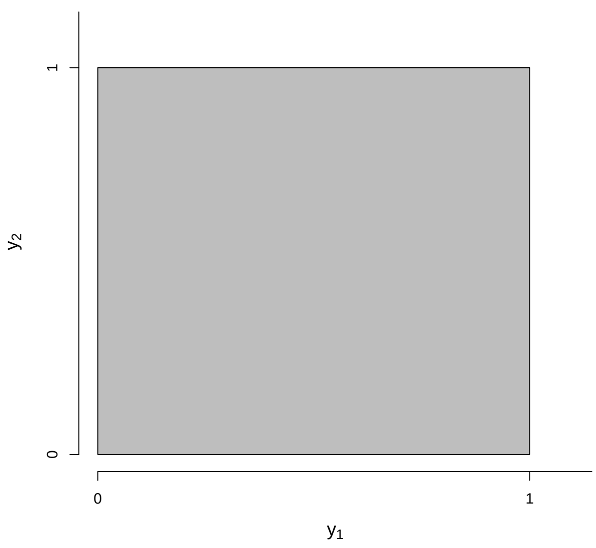
$$f_{Y_1}(y_1) = \begin{cases} 2y_1, & 0 \leq y_1 \leq 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_{Y_2}(y_2) = \begin{cases} 2y_2, & 0 \leq y_2 \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Each marginal pdf matches that of a beta distribution with $\alpha = 2$ and $\beta = 1$; note that

$$\frac{\Gamma(3)}{\Gamma(2)\Gamma(1)} = 2! = 2.$$

In other words, marginally, both Y_1 and Y_2 have a beta distribution with $\alpha = 2$ and $\beta = 1$.

5.60. The support $R = \{(y_1, y_2) : 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1\}$, shown below, does not involve a constraint between y_1 and y_2.



However, we cannot write

$$f_{Y_1,Y_2}(y_1, y_2) = y_1 + y_2 = g(y_1)h(y_2)$$

for nonnegative functions $g(y_1)$ and $h(y_2)$. Therefore, Y_1 and Y_2 are not independent. For fun, let's derive the marginal pdfs and check that indeed

$$f_{Y_1,Y_2}(y_1, y_2) \neq f_{Y_1}(y_1)f_{Y_2}(y_2).$$

For $0 < y_1 < 1$, the marginal pdf of Y_1 is

$$f_{Y_1}(y_1) = \int_{y_2=0}^{1} (y_1 + y_2)\ dy_2 = \left(y_1 y_2 + \frac{y_2^2}{2} \right)\Bigg|_{y_2=0}^{1} = y_1 + \frac{1}{2}.$$

For $0 < y_2 < 1$, the marginal pdf of Y_2 is

$$f_{Y_2}(y_2) = \int_{y_1=0}^{1} (y_1 + y_2)\ dy_1 = \left(\frac{y_1^2}{2} + y_1 y_2 \right)\Bigg|_{y_1=0}^{1} = y_2 + \frac{1}{2}.$$

Summarizing,

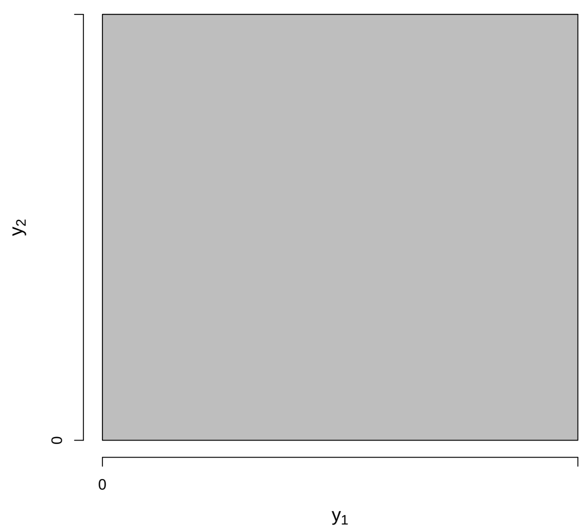
$$f_{Y_1}(y_1) = \begin{cases} y_1 + \frac{1}{2}, & 0 \le y_1 \le 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_{Y_2}(y_2) = \begin{cases} y_2 + \frac{1}{2}, & 0 \le y_2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Clearly,

$$y_1 + y_2 = f_{Y_1,Y_2}(y_1, y_2) \neq f_{Y_1}(y_1) f_{Y_2}(y_2) = \left(y_1 + \frac{1}{2} \right) \left(y_2 + \frac{1}{2} \right).$$

Therefore, Y_1 and Y_2 are not independent.

5.65. First, note the support of Y_1 and Y_2 is $R = \{(y_1, y_2) : y_1 \ge 0, y_2 \ge 0\}$, the entire first quadrant. This set is shown below:



Note that the joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ is a three-dimensional function which takes the value $[1 - \alpha\{(1 - 2e^{-y_1})(1 - 2e^{-y_2})\}]e^{-y_1 - y_2}$ over this region (i.e., the entire first quadrant) and is otherwise equal to zero.

(a) To find the marginal pdf of Y_1, we integrate the joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ over y_2. For $y_1 \ge 0$, this marginal pdf is given by

$$f_{Y_1}(y_1) \quad = \quad \int_{y_2=0}^{\infty} [1 - \alpha\{(1 - 2e^{-y_1})(1 - 2e^{-y_2})\}]e^{-y_1 - y_2}\ dy_2.$$

Let's simplify the integrand algebraically:

$$
\begin{aligned}
& [1-\alpha\{(1-2e^{-y_1})(1-2e^{-y_2})\}]e^{-y_1-y_2} \\
= \; & [1-\alpha(1-2e^{-y_1}-2e^{-y_2}+4e^{-y_1-y_2})]e^{-y_1-y_2} \\
= \; & e^{-y_1-y_2}-\alpha e^{-y_1-y_2}(1-2e^{-y_1}-2e^{-y_2}+4e^{-y_1-y_2}) \\
= \; & e^{-y_1-y_2}-\alpha e^{-y_1-y_2}+2\alpha e^{-y_1}e^{-y_1-y_2}+2\alpha e^{-y_2}e^{-y_1-y_2}-4\alpha e^{-y_1-y_2}e^{-y_1-y_2} \\
= \; & (1-\alpha)e^{-y_1-y_2}+2\alpha e^{-2y_1-y_2}+2\alpha e^{-y_1-2y_2}-4\alpha e^{-2y_1-2y_2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f_{Y_1}(y_1) &= \int_{y_2=0}^{\infty}[(1-\alpha)e^{-y_1-y_2}+2\alpha e^{-2y_1-y_2}+2\alpha e^{-y_1-2y_2}-4\alpha e^{-2y_1-2y_2}]\,dy_2 \\
&= (1-\alpha)\int_{y_2=0}^{\infty}e^{-y_1-y_2}dy_2+2\alpha\int_{y_2=0}^{\infty}e^{-2y_1-y_2}dy_2+2\alpha\int_{y_2=0}^{\infty}e^{-y_1-2y_2}dy_2 \\
&\qquad -4\alpha\int_{y_2=0}^{\infty}e^{-2y_1-2y_2}dy_2.
\end{aligned}
$$

Let's do these integrals in sequence. First,

$$\int_{y_2=0}^{\infty}e^{-y_1-y_2}dy_2 = e^{-y_1}\underbrace{\int_{y_2=0}^{\infty}e^{-y_2}dy_2}_{=1} = e^{-y_1}.$$

Second,

$$\int_{y_2=0}^{\infty}e^{-2y_1-y_2}dy_2 = e^{-2y_1}\underbrace{\int_{y_2=0}^{\infty}e^{-y_2}dy_2}_{=1} = e^{-2y_1}.$$

Third,

$$\int_{y_2=0}^{\infty}e^{-y_1-2y_2}dy_2 = e^{-y_1}\underbrace{\int_{y_2=0}^{\infty}e^{-2y_2}dy_2}_{=1/2} = \frac{1}{2}e^{-y_1}.$$

Fourth,

$$\int_{y_2=0}^{\infty}e^{-2y_1-2y_2}dy_2 = e^{-2y_1}\underbrace{\int_{y_2=0}^{\infty}e^{-2y_2}dy_2}_{=1/2} = \frac{1}{2}e^{-2y_1}.$$

Therefore, for $y_1 \geq 0$,

$$
\begin{aligned}
f_{Y_1}(y_1) &= (1-\alpha)e^{-y_1}+2\alpha e^{-2y_1}+2\alpha\left(\frac{1}{2}\right)e^{-y_1}-4\alpha\left(\frac{1}{2}\right)e^{-2y_1} \\
&= e^{-y_1}-\alpha e^{-y_1}+2\alpha e^{-2y_1}+\alpha e^{-y_1}-2\alpha e^{-2y_1} \\
&= e^{-y_1}.
\end{aligned}
$$

We recognize this as an exponential pdf with mean $\beta = 1$; i.e., $Y_1 \sim$ exponential(1).

(b) An analogous argument will show $Y_2 \sim$ exponential(1). I know this because $f_{Y_1,Y_2}(y_1,y_2)$ is a symmetric function of y_1 and y_2; i.e., $f_{Y_1,Y_2}(y_1,y_2) = f_{Y_1,Y_2}(y_2,y_1)$ and the integral to calculate the marginal pdf of Y_2 is also over $(0,\infty)$.

(c) Suppose $\alpha = 0$. Then, the joint pdf of Y_1 and Y_2 collapses to

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} e^{-y_1-y_2}, & y_1 \geq 0, y_2 \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Note that

$$e^{-y_1-y_2} = f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1)f_{Y_2}(y_2) = e^{-y_1}e^{-y_2}.$$

Therefore, Y_1 and Y_2 are independent. Now suppose Y_1 and Y_2 are independent. If this is true, then there exist functions $g(y_1)$ and $h(y_2)$ such that

$$f_{Y_1,Y_2}(y_1, y_2) = g(y_1)h(y_2).$$

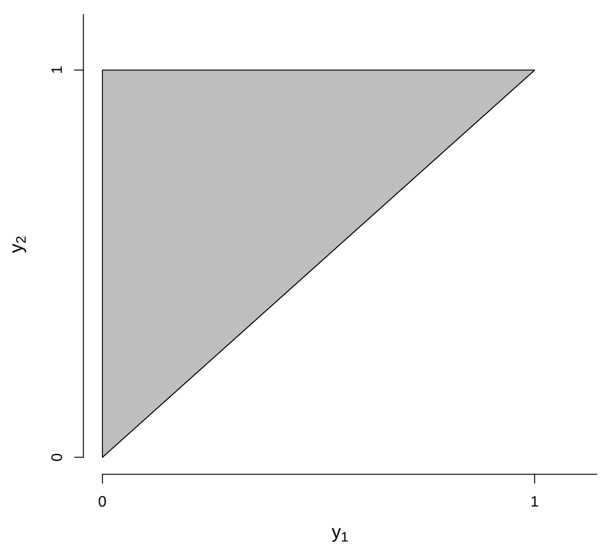
This must be true because the support of Y_1 and Y_2 does not involve a constraint. Therefore, the joint pdf

$$[1 - \alpha\{(1 - 2e^{-y_1})(1 - 2e^{-y_2})\}]e^{-y_1-y_2} = g(y_1)h(y_2),$$

for some functions $g(y_1)$ and $h(y_2)$. The only way $f_{Y_1,Y_2}(y_1, y_2)$ will factor like this is if $\alpha = 0$.

Remark: That $Y_1 \sim$ exponential(1) and $Y_2 \sim$ exponential(1) hold for any value of $\alpha \in [-1, 1]$ is a key feature of this joint pdf. Y_1 and Y_2 will be independent (and hence uncorrelated) if and only if $\alpha = 0$. We can induce correlation between Y_1 and Y_2 by taking $\alpha \neq 0$ and this will not affect the marginal distributions. Interesting!

5.77. The support is $R = \{(y_1, y_2) : 0 \leq y_1 \leq y_2 \leq 1\}$, the upper triangle of the unit square. See below:



The joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ is a three-dimensional function which takes the value $6(1 - y_2)$ over this region and is otherwise equal to zero.

Note: We are being asked to get the marginal means and marginal variances in parts (a) and (b), respectively. Therefore, let's get the marginal distributions first. If these marginal distributions are "named," we might be able to get the means and variances quickly (i.e., by using formulas we have already derived in class).

For $0 < y_1 < 1$, the marginal pdf of Y_1 is

$$\begin{aligned} f_{Y_1}(y_1) = \int_{y_2=y_1}^{1} 6(1-y_2)\, dy_2 &= 6\left(y_2 - \frac{y_2^2}{2}\right)\Big|_{y_2=y_1}^{1} \\ &= 6\left[\left(1-\frac{1}{2}\right) - \left(y_1 - \frac{y_1^2}{2}\right)\right] \\ &= 6\left(\frac{1}{2} - y_1 + \frac{y_1^2}{2}\right) = 3\left(1 - 2y_1 + y_1^2\right) = 3(1-y_1)^2. \end{aligned}$$

For $0 < y_2 < 1$, the marginal pdf of Y_2 is

$$f_{Y_2}(y_2) = \int_{y_1=0}^{y_2} 6(1-y_2)\, dy_1 = 6(1-y_2)\int_{y_1=0}^{y_2} dy_1 = 6y_2(1-y_2).$$

Summarizing,

$$f_{Y_1}(y_1) = \begin{cases} 3(1-y_1)^2, & 0 \le y_1 \le 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_{Y_2}(y_2) = \begin{cases} 6y_2(1-y_2), & 0 \le y_2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Note that

$$\begin{aligned} Y_1 &\sim \text{beta}(1,3) \\ Y_2 &\sim \text{beta}(2,2). \end{aligned}$$

We know the mean and variance for a beta distribution, so now parts (a) and (b) are easy:

(a)

$$E(Y_1) = \frac{1}{1+3} = \frac{1}{4} \quad \text{and} \quad E(Y_2) = \frac{2}{2+2} = \frac{1}{2}.$$

(b)

$$V(Y_1) = \frac{1(3)}{(1+3)^2(1+3+1)} = \frac{3}{80} \quad \text{and} \quad V(Y_2) = \frac{2(2)}{(2+2)^2(2+2+1)} = \frac{4}{80}.$$

(c) Note that

$$E(Y_1 - 3Y_2) = E(Y_1) - 3E(Y_2) = \frac{1}{4} - 3\left(\frac{1}{2}\right) = -\frac{5}{4}.$$

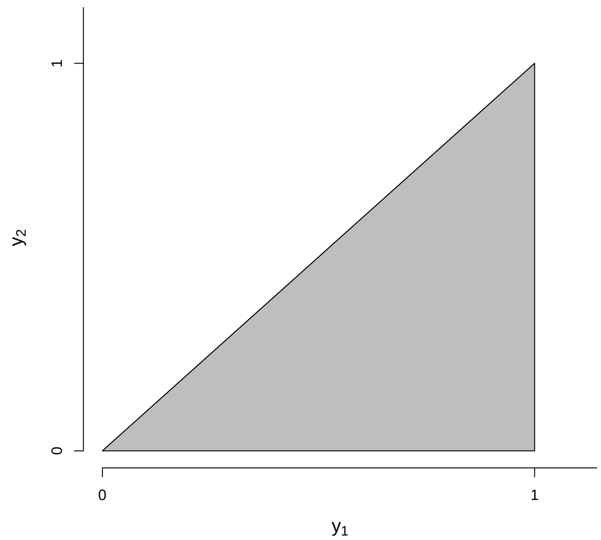
Note: <u>If you didn't derive the marginals first</u>, you could still do this problem, but you would have to do more integration. We can calculate

$$\begin{aligned} E(Y_1) &= \int_{y_1=0}^{1}\int_{y_2=y_1}^{1} y_1\; 6(1-y_2)\, dy_2 dy_1 \\ E(Y_1^2) &= \int_{y_1=0}^{1}\int_{y_2=y_1}^{1} y_1^2\; 6(1-y_2)\, dy_2 dy_1 \end{aligned}$$

and get $V(Y_1)$ using the variance computing formula. Similarly for $E(Y_2)$ and $V(Y_2)$. You could calculate $E(Y_1 - 3Y_2)$ by

$$E(Y_1 - 3Y_2) = \int_{y_1=0}^{1}\int_{y_2=y_1}^{1} (y_1 - 3y_2)\; 6(1-y_2)\, dy_2 dy_1.$$

5.82. The support is $R = \{(y_1, y_2) : 0 \leq y_2 \leq y_1 \leq 1\}$, the lower triangle of the unit square. See below:



The joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ is a three-dimensional function which takes the value $1/y_1$ over this region and is otherwise equal to zero. I think it is probably easiest to calculate $E(Y_1 - Y_2)$ directly; i.e.,

$$\begin{aligned}
E(Y_1 - Y_2) &= \int\int_{\mathbb{R}^2} (y_1 - y_2) f_{Y_1,Y_2}(y_1, y_2)\ dy_1 dy_2 \\
&= \int_{y_2=0}^{1} \int_{y_1=y_2}^{1} (y_1 - y_2) \frac{1}{y_1}\ dy_1 dy_2 \\
&= \int_{y_2=0}^{1} \int_{y_1=y_2}^{1} \left(1 - \frac{y_2}{y_1}\right)\ dy_1 dy_2 \\
&= \int_{y_2=0}^{1} \left[(y_1 - y_2 \ln y_1) \Big|_{y_1=y_2}^{1} \right]\ dy_2 \\
&= \int_{y_2=0}^{1} (1 - y_2 + y_2 \ln y_2)\ dy_2 \ = \int_{y_2=0}^{1} (1 - y_2)\ dy_2 + \int_{y_2=0}^{1} y_2 \ln y_2\ dy_2.
\end{aligned}$$

The first integral is easy; using the beta function result,

$$\int_{y_2=0}^{1} (1 - y_2)\ dy_2 = \frac{\Gamma(1)\Gamma(2)}{\Gamma(3)} = \frac{1}{2}.$$

I did the second integral by parts with

$$\begin{aligned}
u &= \ln y_2 \qquad & du &= \frac{1}{y_2} dy_2 \\
dv &= y_2 & v &= \frac{y_2^2}{2}
\end{aligned}$$

which gives

$$\int_{y_2=0}^{1} y_2 \ln y_2\ dy_2 \ = \ \underbrace{\frac{y_2^2 \ln y_2}{2} \Bigg|_0^1}_{=\,0} - \int_{y_2=0}^{1} \frac{y_2}{2}\ dy_2 = -\frac{1}{2} \left(\frac{y_2^2}{2} \Bigg|_0^1 \right) = -\frac{1}{4}.$$

Therefore,

$$E(Y_1 - Y_2) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Note: I used R to check my work doing the integral

$$\int_{y_2=0}^{1} (1 - y_2 + y_2 \ln y_2)\ dy_2$$

on the preceding page:

```
> integrand = function(x) 1-x+x*log(x)
> integrate(integrand,lower=0,upper=1)
0.2499999 with absolute error < 0.00011
```

5.86. This problem is useful in deriving properties of the t and F distributions (which are ubiquitous in statistical inference). We are given $Z \sim \mathcal{N}(0,1)$,

$$\begin{aligned} Y_1 &\sim \chi^2(\nu_1) \overset{d}{=} \text{gamma}\left(\frac{\nu_1}{2}, 2\right) \\ Y_2 &\sim \chi^2(\nu_2) \overset{d}{=} \text{gamma}\left(\frac{\nu_2}{2}, 2\right), \end{aligned}$$

and Z, Y_1, and Y_2 are mutually independent. I used the notation "$\overset{d}{=}$" which is read "is the same distribution as." Recall that χ^2 distributions are special gamma distributions.

(a) With

$$W = \frac{Z}{\sqrt{Y_1}},$$

we want to derive $E(W)$ and $V(W)$. Because Z and Y_1 are independent, we can write

$$E(W) = E\left(\frac{Z}{\sqrt{Y_1}}\right) = E\left(Z\ \frac{1}{\sqrt{Y_1}}\right) \overset{Z \perp\!\!\!\perp Y_1}{=} E(Z)E\left(\frac{1}{\sqrt{Y_1}}\right) = 0 \times E\left(\frac{1}{\sqrt{Y_1}}\right) = 0,$$

provided that

$$E\left(\frac{1}{\sqrt{Y_1}}\right) < \infty;$$

i.e., this expectation exists. Note that if this expectation does not exist, then we have to deal with a "$0 \times \infty$" situation, which may not be well defined mathematically.

Note that because $Y_1 \sim \text{gamma}(\nu_1/2, 2)$, we can calculate $E(\frac{1}{\sqrt{Y_1}})$ directly. We have

$$\begin{aligned} E\left(\frac{1}{\sqrt{Y_1}}\right) &= \int_0^\infty \frac{1}{\sqrt{y_1}} \underbrace{\frac{1}{\Gamma(\frac{\nu_1}{2})2^{\frac{\nu_1}{2}}}\, y_1^{\frac{\nu_1}{2}-1} e^{-y_1/2}}_{\text{gamma}\left(\frac{\nu_1}{2},2\right)\text{ pdf}}\ dy_1 \\ &= \frac{1}{\Gamma(\frac{\nu_1}{2})2^{\frac{\nu_1}{2}}} \int_0^\infty \underbrace{y_1^{\left(\frac{\nu_1-1}{2}\right)-1} e^{-y_1/2}}_{\text{gamma}\left(\frac{\nu_1-1}{2},2\right)\text{ kernel}}\ dy_1 = \frac{1}{\Gamma(\frac{\nu_1}{2})2^{\frac{\nu_1}{2}}} \times \Gamma\left(\frac{\nu_1 - 1}{2}\right) 2^{\frac{\nu_1-1}{2}} = \frac{\Gamma\left(\frac{\nu_1-1}{2}\right)}{\sqrt{2}\Gamma(\frac{\nu_1}{2})}. \end{aligned}$$

Note that to claim

$$\int_0^\infty y_1^{\left(\frac{\nu_1-1}{2}\right)-1} e^{-y_1/2}\, dy_1 \;=\; \Gamma\left(\frac{\nu_1-1}{2}\right) 2^{\frac{\nu_1-1}{2}}$$

on the preceding page, we need the exponent

$$\frac{\nu_1-1}{2} > 0 \iff \nu_1 > 1.$$

Otherwise, the integral above diverges and $E(\frac{1}{\sqrt{Y_1}})$ does not exist. Therefore,

$$E(W) = E(Z)E\left(\frac{1}{\sqrt{Y_1}}\right) = 0 \times \frac{\Gamma\left(\frac{\nu_1-1}{2}\right)}{\sqrt{2}\Gamma(\frac{\nu_1}{2})} = 0,$$

provided that $\nu_1 > 1$. Note also that the gamma function $\Gamma(\cdot)$ is also defined only for positive arguments, so we can see how the "$\nu_1 > 1$" condition is needed in this way.

Now to calculate $V(W)$, we use the variance computing formula. Note that

$$V(W) = E(W^2) - [E(W)]^2 = E(W^2),$$

because $E(W) = 0$. Therefore, we just need to find the second moment of W. With

$$W = \frac{Z}{\sqrt{Y_1}},$$

we have

$$V(W) = E(W^2) = E\left(\frac{Z^2}{Y_1}\right) = E\left(Z^2\,\frac{1}{Y_1}\right) \overset{Z \perp\!\!\!\perp Y_1}{=} E(Z^2)E\left(\frac{1}{Y_1}\right) = 1 \times E\left(\frac{1}{Y_1}\right) = E\left(\frac{1}{Y_1}\right).$$

Note that because $Z \sim \mathcal{N}(0,1)$, $E(Z) = 0$ and therefore

$$1 = V(Z) = E(Z^2) - [E(Z)]^2 = E(Z^2).$$

Therefore, all we have to do is calculate the first inverse moment $E(\frac{1}{Y_1})$. Again, because $Y_1 \sim \text{gamma}(\nu_1/2, 2)$, we can calculate $E(\frac{1}{Y_1})$ directly.

$$\begin{aligned}
E\left(\frac{1}{Y_1}\right) &= \int_0^\infty \frac{1}{y_1} \underbrace{\frac{1}{\Gamma(\frac{\nu_1}{2})2^{\frac{\nu_1}{2}}}\, y_1^{\frac{\nu_1}{2}-1} e^{-y_1/2}}_{\text{gamma}\left(\frac{\nu_1}{2},2\right)\text{ pdf}}\, dy_1 \\
&= \frac{1}{\Gamma(\frac{\nu_1}{2})2^{\frac{\nu_1}{2}}} \int_0^\infty \underbrace{y_1^{\left(\frac{\nu_1-2}{2}\right)-1} e^{-y_1/2}}_{\text{gamma}\left(\frac{\nu_1-2}{2},2\right)\text{ kernel}}\, dy_1 = \frac{1}{\Gamma(\frac{\nu_1}{2})2^{\frac{\nu_1}{2}}} \times \Gamma\left(\frac{\nu_1-2}{2}\right) 2^{\frac{\nu_1-2}{2}} = \frac{\Gamma\left(\frac{\nu_1-2}{2}\right)}{2\Gamma(\frac{\nu_1}{2})}.
\end{aligned}$$

Analogously to the previous calculation, we need

$$\frac{\nu_1-2}{2} > 0 \iff \nu_1 > 2.$$

Otherwise, the integral above diverges and $E(\frac{1}{Y_1})$ does not exist. We have shown

$$V(W) = E\left(\frac{1}{Y_1}\right) = \frac{\Gamma\left(\frac{\nu_1-2}{2}\right)}{2\Gamma(\frac{\nu_1}{2})} = \frac{\Gamma\left(\frac{\nu_1}{2}-1\right)}{2\Gamma(\frac{\nu_1}{2})} = \frac{\Gamma\left(\frac{\nu_1}{2}-1\right)}{2\left(\frac{\nu_1}{2}-1\right)\Gamma(\frac{\nu_1}{2}-1)} = \frac{1}{2\left(\frac{\nu_1}{2}-1\right)} = \frac{1}{\nu_1-2}.$$

(b) With

$$U = \frac{Y_1}{Y_2},$$

we want to derive $E(U)$ and $V(U)$. Because Y_1 and Y_2 are independent, we can write

$$E(U) = E\left(\frac{Y_1}{Y_2}\right) = E\left(Y_1 \, \frac{1}{Y_2}\right) \overset{Y_1 \perp\!\!\!\perp Y_2}{=} E(Y_1)E\left(\frac{1}{Y_2}\right).$$

We know $E(Y_1) = \nu_1$, the degrees of freedom for $Y_1 \sim \chi^2(\nu_1)$. Therefore, all we have to do is calculate the first inverse moment $E(\frac{1}{Y_2})$. We basically already did this in part (a) when we calculated $E(\frac{1}{Y_1})$. The same argument applies and we get the analogous result; i.e.,

$$E\left(\frac{1}{Y_2}\right) = \frac{\Gamma\left(\frac{\nu_2-2}{2}\right)}{2\Gamma(\frac{\nu_2}{2})} = \frac{1}{\nu_2 - 2}.$$

Therefore, provided $\nu_2 > 2$,

$$E(U) = \frac{\nu_1}{\nu_2 - 2}.$$

Now to calculate $V(U)$, we use the variance computing formula. Note that

$$V(U) = E(U^2) - [E(U)]^2 = E(U^2) - \left(\frac{\nu_1}{\nu_2 - 2}\right)^2.$$

Let's get the second moment of U. With

$$U = \frac{Y_1}{Y_2},$$

we have

$$E(U^2) = E\left(\frac{Y_1^2}{Y_2^2}\right) = E\left(Y_1^2 \, \frac{1}{Y_2^2}\right) \overset{Y_1 \perp\!\!\!\perp Y_2}{=} E(Y_1^2)E\left(\frac{1}{Y_2^2}\right).$$

We can use the variance computing formula to get $E(Y_1^2)$. Note that

$$2\nu_1 = V(Y_1) = E(Y_1^2) - [E(Y_1)]^2 = E(Y_1^2) - \nu_1^2 \implies E(Y_1^2) = 2\nu_1 + \nu_1^2 = \nu_1(2 + \nu_1).$$

Therefore, all we have to do is calculate the second inverse moment $E(\frac{1}{Y_2^2})$. Because $Y_2 \sim$ gamma$(\nu_2/2, 2)$, we can calculate $E(\frac{1}{Y_2^2})$ directly.

$$\begin{aligned} E\left(\frac{1}{Y_2^2}\right) &= \int_0^\infty \frac{1}{y_2^2} \underbrace{\frac{1}{\Gamma(\frac{\nu_2}{2})2^{\frac{\nu_2}{2}}} \, y_1^{\frac{\nu_2}{2}-1} e^{-y_2/2}}_{\text{gamma}\left(\frac{\nu_2}{2},2\right) \text{ pdf}} \, dy_2 \\ &= \frac{1}{\Gamma(\frac{\nu_2}{2})2^{\frac{\nu_2}{2}}} \int_0^\infty \underbrace{y_2^{\left(\frac{\nu_2-4}{2}\right)-1} e^{-y_2/2}}_{\text{gamma}\left(\frac{\nu_2-4}{2},2\right) \text{ kernel}} \, dy_2 = \frac{1}{\Gamma(\frac{\nu_2}{2})2^{\frac{\nu_2}{2}}} \times \Gamma\left(\frac{\nu_2 - 4}{2}\right) 2^{\frac{\nu_2-4}{2}} = \frac{\Gamma\left(\frac{\nu_2-4}{2}\right)}{4\Gamma(\frac{\nu_2}{2})}. \end{aligned}$$

Therefore, for $\nu_2 > 4$,

$$\begin{aligned} E(U^2) = E(Y_1^2)E\left(\frac{1}{Y_2^2}\right) = \frac{\nu_1(2+\nu_1)\Gamma\left(\frac{\nu_2-4}{2}\right)}{4\Gamma(\frac{\nu_2}{2})} &= \frac{\nu_1(2+\nu_1)\Gamma\left(\frac{\nu_2}{2} - 2\right)}{4\Gamma(\frac{\nu_2}{2})} \\ &= \frac{\nu_1(2+\nu_1)}{4\left(\frac{\nu_2}{2} - 1\right)\left(\frac{\nu_2}{2} - 2\right)} = \frac{\nu_1(2+\nu_1)}{(\nu_2 - 2)(\nu_2 - 4)}. \end{aligned}$$

Finally, provided $\nu_2 > 4$,

$$V(U) = E(U^2) - \left(\frac{\nu_1}{\nu_2 - 2}\right)^2 = \frac{\nu_1(2+\nu_1)}{(\nu_2-2)(\nu_2-4)} - \left(\frac{\nu_1}{\nu_2 - 2}\right)^2.$$

This probably simplifies, but I am too tired to try.

5.92. This is the same pdf we examined in Problem 5.77. Recall that we already calculated

$$E(Y_1) = \frac{1}{4} \quad \text{and} \quad E(Y_2) = \frac{1}{2}.$$

Therefore, to find $\text{Cov}(Y_1, Y_2)$, we only have to find $E(Y_1Y_2)$ and then use the covariance computing formula

$$\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2).$$

A picture of the support of Y_1 and Y_2 is given in the solutions to Problem 5.77. We have

$$E(Y_1Y_2) = \int_{y_1=0}^{1}\int_{y_2=y_1}^{1} y_1y_2\ 6(1-y_2)\ dy_2dy_1 = \int_{y_1=0}^{1}\int_{y_2=y_1}^{1} 6y_1y_2(1-y_2)\ dy_2dy_1 = 0.15.$$

I did this double integral numerically in R using the `integral2` function in the `pracma` package:

```
> library(pracma)
> integrand <- function(y1,y2) 6*y1*y2*(1-y2)
> y2min <- function(y1) y1
> integral2(integrand,0,1,y2min,1)
$`Q`
[1] 0.15
$error
[1] 1.387779e-17
```

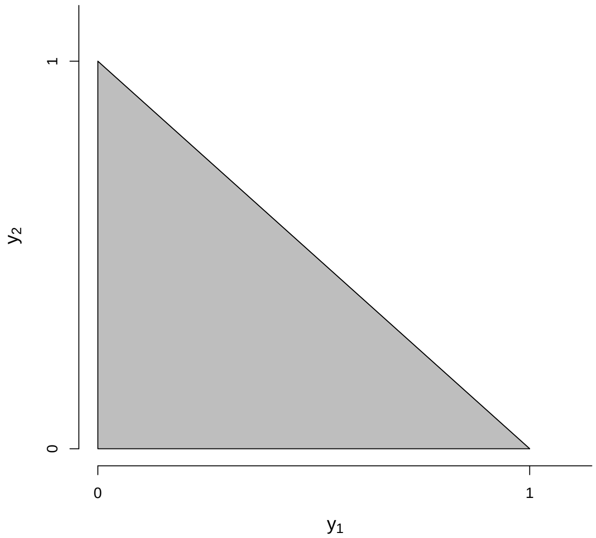
To get this "by hand," note that the last double integral equals

$$\begin{aligned}
\int_{y_1=0}^{1} 6y_1\left[\int_{y_2=y_1}^{1}(y_2 - y_2^2)\ dy_2\right]dy_1 &= \int_{y_1=0}^{1} 6y_1\left(\frac{y_2^2}{2} - \frac{y_2^3}{3}\right)\Bigg|_{y_2=y_1}^{1} dy_1 \\
&= \int_{y_1=0}^{1} 6y_1\left(\frac{1}{6} - \frac{y_1^2}{2} + \frac{y_1^3}{3}\right)dy_1 \\
&= \int_{y_1=0}^{1} (y_1 - 3y_1^3 + 2y_1^4)\ dy_1 \\
&= \left(\frac{y_1^2}{2} - \frac{3y_1^4}{4} + \frac{2y_1^5}{5}\right)\Bigg|_{y_1=0}^{1} = \frac{1}{2} - \frac{3}{4} + \frac{2}{5} = 0.15.
\end{aligned}$$

Therefore, the covariance of Y_1 and Y_2 is

$$\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = 0.15 - \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = 0.025,$$

which indicates a positive linear relationship between Y_1 and Y_2. Because $\text{Cov}(Y_1, Y_2) \neq 0$, we know that Y_1 and Y_2 are not independent (i.e., they are dependent). We could have also quickly concluded this from the triangular support of Y_1 and Y_2.



5.107. The support $R = \{(y_1, y_2) : 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, y_1 + y_2 \leq 1\}$ is the triangular region in the picture above. Note that the upper boundary of this support is

$$y_1 + y_2 = 1 \implies y_2 = 1 - y_1,$$

a linear function of y_1 with slope -1 and intercept 1. The joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ is a three-dimensional function which takes the value 2 over this region and is otherwise equal to zero. In other words, the joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ is **constant** (with height 2) over this triangle.

To calculate $E(Y_1 + Y_2)$, we could work directly with the joint pdf and calculate

$$E(Y_1 + Y_2) = \int_{y_1=0}^{1} \int_{y_2=0}^{1-y_1} 2(y_1 + y_2)\ dy_2 dy_1 = \frac{2}{3}.$$

Alternatively, you could get the marginal pdfs $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$, get the marginal means from those, and then calculate $E(Y_1 + Y_2) = E(Y_1) + E(Y_2)$; see the end of this solution. I did it the way above because it is just as easy. I used R to do the double integral above using the `integral2` function in the `pracma` package:

```
> library(pracma)
> integrand <- function(y1,y2) 2*(y1+y2)
> y2max <- function(y1) 1-y1
> integral2(integrand,0,1,0,y2max)
$`Q`
[1] 0.6666667
$error
[1] 1.387779e-17
```

We can also do this integral by hand; note that

$$\begin{aligned}
E(Y_1+Y_2) &= \int_{y_1=0}^{1}\int_{y_2=0}^{1-y_1} 2(y_1+y_2)\ dy_2dy_1 \\
&= 2\int_{y_1=0}^{1}\left[\left(y_1y_2+\frac{y_2^2}{2}\right)\Big|_{y_2=0}^{1-y_1}\right]\ dy_1 \\
&= 2\int_{y_1=0}^{1}\left[y_1(1-y_1)+\frac{(1-y_1)^2}{2}\right]\ dy_1 \\
&= 2\int_{y_1=0}^{1} y_1(1-y_1)dy_1+\int_{y_1=0}^{1}(1-y_1)^2dy_1 = \frac{2\Gamma(2)\Gamma(2)}{\Gamma(4)}+\frac{\Gamma(1)\Gamma(3)}{\Gamma(4)} = \frac{1}{3}+\frac{1}{3}=\frac{2}{3}.
\end{aligned}$$

Now, to get $V(Y_1+Y_2)$, we have options.

Option 1: Use the variance computing formula on the random variable Y_1+Y_2; i.e.,

$$V(Y_1+Y_2)=E[(Y_1+Y_2)^2]-[E(Y_1+Y_2)]^2=E[(Y_1+Y_2)^2]-\left(\frac{2}{3}\right)^2.$$

The second moment of Y_1+Y_2 is

$$E[(Y_1+Y_2)^2]=\int_{y_1=0}^{1}\int_{y_2=0}^{1-y_1} 2(y_1+y_2)^2\ dy_2dy_1=\frac{1}{2}.$$

I used R to do the double integral above using the `integral2` function in the `pracma` package:

```
> library(pracma)
> integrand <- function(y1,y2) 2*(y1+y2)^2
> y2max <- function(y1) 1-y1
> integral2(integrand,0,1,0,y2max)
$`Q`
[1] 0.5
$error
[1] 5.551115e-17
```

Therefore,

$$V(Y_1+Y_2)=E[(Y_1+Y_2)^2]-[E(Y_1+Y_2)]^2=\frac{1}{2}-\left(\frac{2}{3}\right)^2=\frac{1}{18}.$$

Option 2: This will take longer; it utilizes the formula

$$V(Y_1+Y_2)=V(Y_1)+V(Y_2)+2\text{Cov}(Y_1,Y_2).$$

Let's get the marginal pdfs $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$ after all. For $0<y_1<1$, the marginal pdf of Y_1 is

$$f_{Y_1}(y_1)=\int_{y_2=0}^{1-y_1} 2\ dy_2\ =\ 2(1-y_1).$$

For $0 < y_2 < 1$, the marginal pdf of Y_2 is

$$f_{Y_2}(y_2) = \int_{y_1=0}^{1-y_2} 2 \, dy_1 \;=\; 2(1-y_2).$$

Summarizing,

$$f_{Y_1}(y_1) = \begin{cases} 2(1-y_1), & 0 \le y_1 \le 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_{Y_2}(y_2) = \begin{cases} 2(1-y_2), & 0 \le y_2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Note that

$$\begin{aligned} Y_1 &\sim \text{beta}(1,2) \\ Y_2 &\sim \text{beta}(1,2). \end{aligned}$$

Therefore,

$$V(Y_1) = V(Y_2) = \frac{1(2)}{(1+2)^2(1+2+1)} = \frac{1}{18}.$$

Now, to get $\text{Cov}(Y_1, Y_2)$, we will use the covariance computing formula:

$$\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2).$$

Because $Y_1 \sim \text{beta}(1,2)$ and $Y_2 \sim \text{beta}(1,2)$, we know

$$E(Y_1) = E(Y_2) = \frac{1}{1+2} = \frac{1}{3}.$$

Also,

$$E(Y_1Y_2) = \int_{y_1=0}^{1} \int_{y_2=0}^{1-y_1} 2y_1y_2 \, dy_2 dy_1 = \frac{1}{12}.$$

I used R to do the double integral above using the `integral2` function in the `pracma` package:

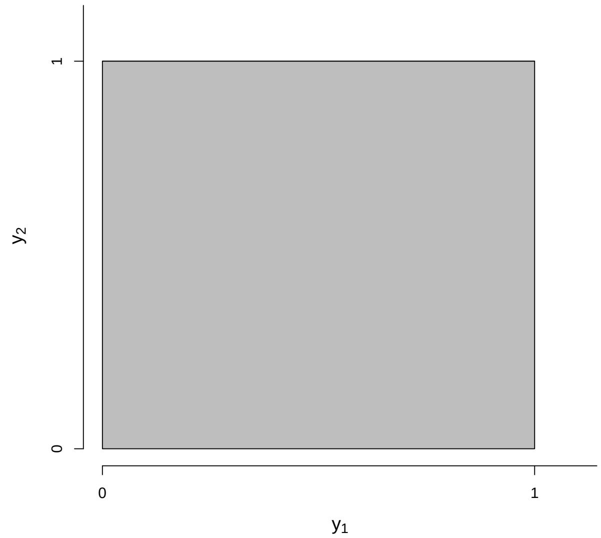
```
> library(pracma)
> integrand <- function(y1,y2) 2*y1*y2
> y2max <- function(y1) 1-y1
> integral2(integrand,0,1,0,y2max)
$‘Q‘
[1] 0.08333333
$error
[1] 7.806256e-18
```

Therefore,

$$\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = \frac{1}{12} - \frac{1}{3}\left(\frac{1}{3}\right) = -\frac{1}{36}.$$

Therefore,

$$V(Y_1 + Y_2) = V(Y_1) + V(Y_2) + 2\text{Cov}(Y_1, Y_2) = \frac{1}{18} + \frac{1}{18} + 2\left(-\frac{1}{36}\right) = \frac{1}{18}.$$



5.109. The support $R = \{(y_1, y_2) : 0 \le y_1 \le 1, 0 \le y_2 \le 1\}$ is the unit square in the picture above. The joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ is a three-dimensional function which takes the value $y_1 + y_2$ over this region and is otherwise equal to zero.

Note: We worked with this joint pdf in Problem 5.60. We derived the marginal pdfs to be

$$f_{Y_1}(y_1) = \begin{cases} y_1 + \frac{1}{2}, & 0 \le y_1 \le 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_{Y_2}(y_2) = \begin{cases} y_2 + \frac{1}{2}, & 0 \le y_2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Let's get the marginal means (they are identical). We have

$$E(Y_1) = \int_{y_1=0}^{1} y_1 \left(y_1 + \frac{1}{2} \right) dy_1 = \left(\frac{y_1^3}{3} + \frac{y_1^2}{4} \right) \Bigg|_{y_1=0}^{1} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

The same calculation shows $E(Y_2) = \frac{7}{12}$ also. Therefore,

$$E(30Y_1 + 25Y_2) = 30E(Y_1) + 25E(Y_2) = 30 \left(\frac{7}{12} \right) + 25 \left(\frac{7}{12} \right) \approx 32.083.$$

The problem asks us to find $V(30Y_1 + 25Y_2)$. This is

$$V(30Y_1 + 25Y_2) = 30^2 V(Y_1) + 25^2 V(Y_2) + 2(30)(25)\text{Cov}(Y_1, Y_2).$$

Let's get the marginal variances (they are identical). The second moment of Y_1 is

$$E(Y_1^2) = \int_{y_1=0}^{1} y_1^2 \left(y_1 + \frac{1}{2} \right) dy_1 = \left(\frac{y_1^4}{4} + \frac{y_1^3}{6} \right) \Bigg|_{y_1=0}^{1} = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}.$$

Therefore, the marginal variance of Y_1 is

$$V(Y_1) = E(Y_1^2) - [E(Y_1)]^2 = \frac{5}{12} - \left(\frac{7}{12} \right)^2 = \frac{11}{144}.$$

The same calculation shows $V(Y_2) = \frac{11}{144}$ also. Now, let's get the covariance. We will use the covariance computing formula:

$$\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2).$$

We already know $E(Y_1) = E(Y_2) = \frac{7}{12}$. Also,

$$E(Y_1Y_2) = \int_{y_1=0}^{1}\int_{y_2=0}^{1} y_1y_2(y_1+y_2)\ dy_2dy_1 = \frac{1}{3}.$$

I used R to do the double integral above using the `integral2` function in the `pracma` package:

```
> library(pracma)
> integrand <- function(y1,y2) y1*y2*(y1+y2)
> integral2(integrand,0,1,0,1)
$‘Q‘
[1] 0.3333333
$error
[1] 2.775558e-17
```

Therefore,

$$\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = \frac{1}{3} - \frac{7}{12}\left(\frac{7}{12}\right) = -\frac{1}{144}.$$

Finally,

$$\begin{aligned} V(30Y_1 + 25Y_2) &= 30^2V(Y_1) + 25^2V(Y_2) + 2(30)(25)\text{Cov}(Y_1, Y_2) \\ &= 900\left(\frac{11}{144}\right) + 625\left(\frac{11}{144}\right) + 2(30)(25)\left(-\frac{1}{144}\right) \approx 106.076. \end{aligned}$$

Summary: We have a random variable $30Y_1 + 25Y_2$ with mean and variance

$$\begin{aligned} \mu &\approx 32.083 \\ \sigma^2 &\approx 106.076. \end{aligned}$$

We don't know the distribution of the random variable $30Y_1+25Y_2$, so we can't make probability calculations exactly. We can use Tchebysheff's Theorem to say

$$P(\mu - k\sigma < 30Y_1 + 25Y_2 < \mu + k\sigma) \geq 1 - \frac{1}{k^2},$$

with the values of μ and σ^2 above. If we take $k = 2$, we get

$$P(\mu - 2\sigma < 30Y_1 + 25Y_2 < \mu + 2\sigma) \geq 1 - \frac{1}{2^2} = 0.75.$$

The lower endpoint is

$$32.083 - 2\sqrt{106.076} \approx 11.484$$

and the upper endpoint is

$$32.083 + 2\sqrt{106.076} \approx 52.681.$$

Therefore, the total productivity $30Y_1+25Y_2$ will fall between 11.484 and 52.681 with probability at least 0.75.